Homology Theories With Cubical Bars

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Abstract

Here a new (so we hope) class of homology theories will be presented. The foundation is the 'classical' singular cubical homology theory, developed in [1]. We generalize this by 'drawing' in each dimension a 'bar' on the cube. This will be taken as the boundary. We get a chain complex, hence homology modules. With additional conditiones the homotopy axiom and the excision axiom hold, and finally we have an extraordinary homology theory. By dividing it with a suitable quotient module we get the ordinary singular homology theory.

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1 Introduction

For a better understanding of this paper it would be very usefull that the reader have some knowledge about the cubical singular homology theory, how, for instance, it is constructed in [1]. Because this construction is only one of many possible ways to get the well known singular homology theory, it seems that today mathematicians have only historical interests in it, but no mathematical interest, because by using simplices instead of cubes one gets isomorphic homology groups, and the simplicial homology theory as it is presented in [7] is well understand and a common tool of topologists. In this book [7] is also is been shown, that a homology theory is uniquely defined by only four famous properties, the 'Eilenberg-Steenrod Axioms'. This singular homology theory is a very usefull and successfull method of the mathematicians, as well as there are applications in other fields of science, even the cubical variant is used, for instance for digital image processing and nonlinear dynamics, see [8].

Here we show that there is an easy way to generalize cubical singular homology theory. In the ordinary cubical singular homology theory the boundary operator is constructed by taking the topological boundary of a n-dimensional cube as a linear combination of $2{\cdot}n$ cubes of dimension $(n-1)$, provided with alternating signs. In this paper this way is generalized by 'drawing' in all n directions a fixed linear combination of $(n-1)$-dimensional cubes.

Because we construct this for every n, we get a chain complex with falling dimensions. With some additional conditions, one can show that the homotopy axiom and the excision axiom hold, and as a result we get an extraordinary homology theory, and after all it turns out that we have constructed a generalization of the singular homology theory.

A strong disadvantage is that, for a given space X, either we are not able to compute the homology moduls or we can compute them by expressing them using the ordinary singular homology theory. (This clever method is described in [3]). Hence with this new theory we only can get informations which we already are able to get with the usual singular homology theory. Because of that, our construction may have no practical use, but perhaps there is a theoretical interest.

The results expressed in a more formal way:

For all $L \in \mathbb{N}$, for every $(L+1)$-tuple $\vec{m} := [m_0, m_1, m_2, \dots, m_L] \in \mathbb{Z}^{L+1}$ for all $n \in \mathbb{N}_0$ a functor ${}_{\vec{m}}\mathcal{H}_n : \mathbf{TOP^2}$ to $\mathbf{AB}$ will be constructed.

The homotopy axiom holds if and only if the greatest common divisor of $\{m_0, m_1, m_2, \dots, m_L\}$ is 1 . If $L = 1$ and if $\gcd\{m_0, m_1\} = 1$ the excision axiom holds, and by dividing it with a suitable quotient module, (generated with 'degenerate' maps) we will get the ordinary singular homology theory.

As we mentioned above, the knowledge of [1] (specially the pages 11-37) will make the following considerations much easier, because whenever it was possible, at our constructions we followed the lines of Prof. Massey.

2 Some Definitions

In this paper we use the customary notations: $\mathbb{N} := \{1, 2, 3, \dots\}$, $\mathbb{N}_0 := \{0, 1, 2, 3, \dots\}$, $\mathbb{N}_{-1} := \{-1, 0, 1, 2, 3, \dots\}$, $\mathbb{Z} := \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$, and $\mathbb{R}$ for the real numbers. The expression $A \hookrightarrow B$ means (depending on the circumstances) 'B is a topological space and A is a subspace' or 'A is a subgroup of B' or 'A is a submodule of B'.

The brackets $(\cdots)$ and $[\cdots]$ we'll use for structuring text and formulas, $[\cdots]$ also for intervals and tuples. The brackets $\langle\cdots\rangle$ will be needed for the boundary operator, $\|\cdots\|$ for the subdivision operator and $\{\cdots\}$ for sets.

Definition 1. Let for all $n \in \mathbb{N}$ $W^n := \{ (x_1, x_2, x_3, \ldots, x_n) \in \mathbb{R}^n \mid$ for all $i \in \{ 1, 2, 3, \ldots, n\} \Rightarrow x_i \in [0,1] \}$, and $W^0 := \{ 0 \}$. Let $\mathcal{R}$ be a commutative ring with unit 1 . Let
$\mathcal{CC}_{n,X} := \{T : W^n \to X \mid T$ is continuous $\}$, $\Phi\left[(\mathcal{R})\,\mathcal{CC}\right]_n(X) :=$ free $\mathcal{R}$-module with the base $\mathcal{CC}_{n,X}$, and $\Phi\left[(\mathcal{R})\,\mathcal{CC}\right]_{-1}(X) := 0$, the trivial $\mathcal{R}$-module.
As further abbreviations take $\Phi_{\mathcal{R},n}(X) := \Phi\left[(\mathcal{R})\,\mathcal{CC}\right]_n(X)$, and with $\mathcal{R} := \mathbb{Z}$ let $\Phi_n(X) := \Phi_{\mathbb{Z},n}(X) := \Phi\left[(\mathbb{Z})\,\mathcal{CC}\right]_n(X)$.
Every $\mathrm{u} \in \Phi\left[(\mathcal{R})\,\mathcal{CC}\right]_n(X)$ is called a ' chain '.

That means that W^n is the n-dimensional standardcube with the usual euclidian topology. One has $W^1 = [\ 0, 1\ ]$, the unit interval. $\mathcal{CC}$ means ' Continuous Cubes '. Moreover all maps we'll use are continuous.

Definition 2. Let $\mathbf{TOP^2}$ be the category of pairs of topological spaces and continuous maps as morphisms. Let $\mathbf{AB}$ be the category of abelian groups and their morphisms. Let $\mathcal{R}$-$\mathbf{MOD}$ be the category of $\mathcal{R}$-Moduls.

That means that $f : (X, A) \to (Y, B) \in \mathbf{TOP^2}$ if and only if X and Y are topological spaces and $A \subset X$, $B \subset Y$ and A, B are carrying the subspace topology and f is continuous and $f(A) \subset B$. To prepare the following steps we make a brief preview of the coming constructions.

To define the homology theory $\mathcal{H}$ we will create a sequence of functors $(\Phi_n)_{n \in \mathbb{N}_{-1}}$, Φ_n: $\mathbf{TOP^2} \to \mathcal{R}$-$\mathbf{MOD}$, and a sequence of natural transformations: $(\ \partial_n : (\Phi_n) \to (\Phi_{n-1})\ )_{n \in \mathbb{N}_0}$, which will be called the boundary operator, with the following properties: For $f : (X, A) \to (Y, B) \in \mathbf{TOP^2}$ for every $n \in \mathbb{N}_0$ the diagram

$$
\begin{array}{ccccccc}
\ldots & \xrightarrow{\partial_{n+2}} \Phi_{n+1}(X,A) & \xrightarrow{\partial_{n+1}} & \Phi_n(X,A) & \xrightarrow{\partial_n} & \Phi_{n-1}(X,A) \xrightarrow{\partial_{n-1}} & \ldots \\
\ldots & \downarrow \Phi_{n+1}(f) & & \downarrow \Phi_n(f) & & \downarrow \Phi_{n-1}(f) & \ldots \\
\ldots & \xrightarrow{\partial_{n+2}} \Phi_{n+1}(Y,B) & \xrightarrow{\partial_{n+1}} & \Phi_n(Y,B) & \xrightarrow{\partial_n} & \Phi_{n-1}(Y,B) \xrightarrow{\partial_{n-1}} & \ldots
\end{array}
$$

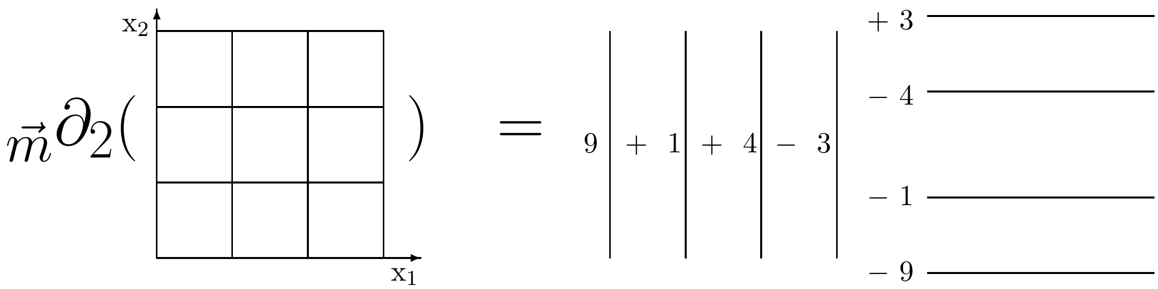
will commute, i. e. : $\Phi_{n-1}(f) \circ \partial_n = \partial_n \circ \Phi_n(f)$.
Moreover for all $n \in \mathbb{N}_0$ will hold that $\partial_n \circ \partial_{n+1} = 0$. Then we will be able to define for all $n \in \mathbb{N}_0$ the $\mathcal{R}$-Modul : $\mathcal{H}_n(X, A) := \frac{kernel(\partial_n)}{image(\partial_{n+1})}$, and this will lead for all $n \in \mathbb{N}_0$ to a functor $\mathcal{H}_n$: $\mathbf{TOP^2} \longrightarrow \mathcal{R}$-$\mathbf{MOD}$. $\mathcal{H}_n(X, A)$ will be called the n^{th} homology module of the pair (X, A).

3 The Boundary Operator

Now we define for all $L \in \mathbb{N}$ for all tuples $\vec{m} = [m_0, m_1, m_2, \ldots, m_L] \in \mathcal{R}^{L+1}$ for all $n \in \mathbb{N}_0$ a ' boundary operator ' : ${}_{\vec{m}}\partial_n : \Phi\left[(\mathcal{R})\,\mathcal{CC}\right]_n(X) \to \Phi\left[(\mathcal{R})\,\mathcal{CC}\right]_{n-1}(X)$, or, briefly ${}_{\vec{m}}\partial_n : \Phi_{\mathcal{R},n}(X) \to \Phi_{\mathcal{R},n-1}(X)$.
Let for all $n \in \mathbb{N}\backslash\{1\}$ for all $T \in \mathcal{CC}_{n,X}$ for all $i \in \{0,1,2, \ldots , L\}$ and for all $j \in \{1, 2, 3, \ldots, n\}$: $\langle T \rangle_{n,\, i,\, j}$ be an element out of $\mathcal{CC}_{n-1,X}$ by defining:
For every tuple $(\ x_1, x_2, x_3, \ldots , x_{n-1}\ ) \in W^{n-1}$ let
$\langle T \rangle_{n,\, i,\, j}\ (\ x_1, x_2, x_3, \ldots , x_{n-1}\ ) := T\ (\ x_1, x_2, x_3, \ldots , x_{j-1}, \frac{i}{L}, x_j, \ldots , x_{n-1}\ )$.
For $n = 1$ let $\langle T \rangle_{1,\, i,\, 1} \in \mathcal{CC}_{0,X}$ by defining $\langle T \rangle_{1,\, i,\, 1}\ (0) := T(\frac{i}{L}\ )$.
Finally let for every $n \in \mathbb{N}$:

$$ {}_{\vec{m}}\partial_n(T) := \sum_{j=1}^{n} (-1)^{j+1} \sum_{i=0}^{L} m_i \cdot \langle T \rangle_{\,n,\,i,\,j} \tag{1} $$

and for $n = 0$ let ${}_{\vec{m}}\partial_0(T) := 0$, the only possible map.
See the picture which demonstrates the case $n = 2$, $L = 3$ and $\vec{m}$ = [9,1,4,−3] :



For a 'chain' u $= r_1\ T_1 + r_2\ T_2 \in \Phi_{\mathcal{R},n}(X)$ let ${}_{\vec{m}}\partial_n$(u) be defined by linear extension:
${}_{\vec{m}}\partial_n(r_1\ T_1 + r_2\ T_2) := r_1\ {}_{\vec{m}}\partial_n\ (\ T_1\ ) + r_2\ {}_{\vec{m}}\partial_n\ (\ T_2\ )$.

Remark: It is trivial that W^n is an n-dimensional manifold with boundary. But for $L \geq 2$ the map ${}_{\vec{m}}\partial_n$ is defined not only on the topological boundary, but also on parts of the interior of W^n. The name ' boundary operator ' we only use for historical reasons.

Theorem 1. *For all* $L \in \mathbb{N}$ *for all* $(L+1)$*-tuples* $\vec{m} := [m_0, m_1, m_2, \ldots, m_L] \in \mathcal{R}^{L+1}$ *for all* $n \in \mathbb{N}$ *we have* ${}_{\vec{m}}\partial_{n-1} \circ {}_{\vec{m}}\partial_n = 0$ *.*

Proof. For $n = 1$ the statement is trivial. For $n = 2$ the proof is similar to the cases with $n \geq 3$ which we will show in details, only sometimes a little bit easier, so we will cut out this case. Thus, let $n \geq 3$.
Because of the linearity of ${}_{\vec{m}}\partial_n$ it suffices to prove the theorem on the base of $\Phi_{\mathcal{R},n}(X)$. Hence let $T \in \mathcal{CC}_{n,X}$, that means $T : W^n \to$ X, and T is continuous.
We have

$$ \begin{aligned} {}_{\vec{m}}\partial_{n-1} \circ {}_{\vec{m}}\partial_n(T) &= {}_{\vec{m}}\partial_{\,n-1}\ (\sum_{j=1}^{n} (-1)^{j+1} \sum_{i=0}^{L} m_i \cdot \langle T \rangle_{\,n,\,i,\,j}\ ) \\ &= \sum_{j=1}^{n} (-1)^{j+1} \sum_{i=0}^{L} m_i \cdot\ \ {}_{\vec{m}}\partial_{n-1}(\langle T \rangle_{\,n,\,i,\,j}\ ) \\ &= \sum_{j=1}^{n} (-1)^{j+1} \sum_{i=0}^{L} m_i \cdot \sum_{p=1}^{n-1} (-1)^{p+1} \sum_{k=0}^{L} m_k \cdot \Big\langle \langle T \rangle_{\,n,\,i,\,j} \Big\rangle_{\,n-1,\,k,\,p} \end{aligned} $$

Hence one gets :

$$ {}_{\vec{m}}\partial_{n-1} \circ {}_{\vec{m}}\partial_n(T) = \sum_{j=1}^{n} \sum_{p=1}^{n-1} (-1)^{j+p+2} \cdot \sum_{i=0}^{L} \sum_{k=0}^{L} m_i \cdot m_k \cdot \Big\langle \langle T \rangle_{\,n,\,i,\,j} \Big\rangle_{\,n-1,\,k,\,p} \tag{2} $$

Now let $(\ x_1,\ x_2,\ x_3,\ \ldots\ , x_{n-2}\ )$ be an arbitrary element out of W^{n-2} , and for fixed

$i, k \in \{0, 1, 2, \ldots, L\}$ holds that

$$\begin{aligned}
& \Big\langle \langle T \rangle_{\,n,\,i,\,j} \Big\rangle_{\,n-1,\,k,\,p} \ (\ (x_1,\ x_2,\ x_3,\ \ldots\ , x_{n-2}\ )\ ) \\
= \ & \langle T \rangle_{\,n,\,i,\,j} \ (\ (x_1,\ x_2,\ x_3,\ \ldots\ ,\ x_{p-1},\ \frac{k}{L}\ ,\ x_p\ ,\ \ldots\ ,\ x_{n-2}\ )\ ) \\
= \ & \begin{cases} T(\ x_1,\ x_2,\ x_3,\ \ldots\ ,\ x_{j-1},\ \frac{i}{L},\ x_j,\ x_{j+1}, \ldots,\ x_{p-1},\ \frac{k}{L}\ ,\ x_p\ ,\ \ldots\ ,\ x_{n-2}\ ) & if\ j \le p \\ T(\ x_1,\ x_2,\ x_3,\ \ldots\ ,\ x_{p-1},\ \frac{k}{L},\ x_p,\ x_{p+1}, \ldots,\ x_{j-2},\ \frac{i}{L}\ ,\ x_{j-1},\ x_j, \ldots\ ,\ x_{n-2}\ ) & if\ j > p \end{cases}
\end{aligned}$$

The sign depends on j and p only. Hence we have for $j \le p$:
$(-1)^{j+p+2}\ \ m_i\, m_k\ \Big\langle \langle T \rangle_{\,n,\,i,\,j} \Big\rangle_{\,n-1,\,k,\,p} \ + \ (-1)^{j+(p+1)+2}\ \ m_k\, m_i\ \Big\langle \langle T \rangle_{\,n,\,k,\,p+1} \Big\rangle_{\,n-1,\,i,\,j} = 0$
The set $M := \{1, 2, 3, \ldots, n\} \times \{1, 2, 3, \ldots, n-1\}$ contains $n \cdot (n-1)$ elements. With $M_{small} := \{(j,p) \in M \mid j \le p\}$ and $M_{big} := \{(j,p) \in M \mid j > p\}$ we have $M = M_{small} \cup M_{big}$, and $M_{small} \cap M_{big} = \emptyset$. The map $M_{small} \to M_{big}$, $(j,p) \ \mapsto \ (p+1, j)$ is bijective. Thus the $n(n-1)(L+1)^2$ functions in (2) eliminate themselves two by two.
$\Longrightarrow {}_{\vec{m}}\partial_{n-1} \ \circ \ {}_{\vec{m}}\partial_n(T) = 0$. □

For each topological space we get a so-called 'chain complex' ${}_{\vec{m}}\mathcal{K}(\mathrm{X})$, that means a sequence of $\mathcal{R}$-module homomorphisms $({}_{\vec{m}}\partial_n)_{n \in \mathbb{N}_0}$. In details:

$${}_{\vec{m}}\mathcal{K}(X) \ := \ \ldots \xrightarrow{{}_{\vec{m}}\partial_{n+1}} \ \Phi_{\mathcal{R},n}(X) \xrightarrow{{}_{\vec{m}}\partial_n} \ \Phi_{\mathcal{R},n-1}(X) \xrightarrow{{}_{\vec{m}}\partial_{n-1}} \ \Phi_{\mathcal{R},n-2}(X) \xrightarrow{{}_{\vec{m}}\partial_{n-2}} \ \ldots \qquad (3)$$

Because of ${}_{\vec{m}}\partial_n \ \circ \ {}_{\vec{m}}\partial_{n+1} \ = \ 0$ the $\mathcal{R}$-module

$${}_{\vec{m}}\mathcal{H}_n(X) := \frac{kernel({}_{\vec{m}}\partial_n)}{image({}_{\vec{m}}\partial_{n+1})} \qquad (4)$$

is well defined for all topological spaces X and all $\vec{m} \in \mathcal{R}^{L+1}$, $n \in \mathbb{N}_0$!

Definition 3. *The (L+1) tuple* $\vec{m} = [m_0, m_1, m_2, \ldots, m_L]$ *is called the 'weight' of the homology theory, the number* $\sigma := \sum_{i=0}^{L} m_i \in \mathcal{R}$ *is the 'index',* $L \in \mathbb{N}$ *is the 'length' of this homology theory. An* $u \in kernel\ (\ {}_{\vec{m}}\partial_n\ )$ *is called a 'cycle', an* $w \in image\ (\ {}_{\vec{m}}\partial_{n+1}\ )$ *is called a 'boundary'.*

Example :
Let $\mathcal{R} := \mathbb{Z}$. For the one-point space $\{p\}$ and for $n \in \mathbb{N}_0$ there is only one T: $W^n \to \{p\}$, thus it holds $\Phi_n(p) \cong \mathbb{Z}$. And as the chain complex (3)

$${}_{\vec{m}}\mathcal{K}(p) \ \ := \ \ \ldots \xrightarrow{{}_{\vec{m}}\partial_4} \Phi_3(p) \xrightarrow{{}_{\vec{m}}\partial_3} \Phi_2(p) \xrightarrow{{}_{\vec{m}}\partial_2} \Phi_1(p) \xrightarrow{{}_{\vec{m}}\partial_1} \Phi_0(p) \xrightarrow{{}_{\vec{m}}\partial_0} \{0\} \qquad (5)$$

we get ${}_{\vec{m}}\mathcal{K}(p) = \ldots \xrightarrow{{}_{\vec{m}}\partial_4} \mathbb{Z} \xrightarrow{{}_{\vec{m}}\partial_3} \mathbb{Z} \xrightarrow{{}_{\vec{m}}\partial_2} \mathbb{Z} \xrightarrow{{}_{\vec{m}}\partial_1} \mathbb{Z} \xrightarrow{{}_{\vec{m}}\partial_0} \{0\}$.
The boundary operators are

$${}_{\vec{m}}\partial_n = \begin{cases} 0 & for\ n\ even \\ \times\sigma & for\ n\ odd \end{cases}$$

with $\times\sigma$: $\mathbb{Z} \to \mathbb{Z}$, $\mathrm{x} \mapsto \sigma \cdot \mathrm{x}$.
Explanation : Because of the alternating sign in the definition of
${}_{\vec{m}}\partial_n(T) := \sum_{j=1}^{n} (-1)^{j+1} \sum_{i=0}^{L} m_i \ \langle T \rangle_{\,n,\,i,\,j}$ always σ of these maps are eliminating themselves. That means for $\sigma \neq 0$

$${}_{\vec{m}}\mathcal{H}_n(p) = \begin{cases} 0 & for\ n\ odd \\ \mathbb{Z}_\sigma & for\ n\ even\ , \end{cases}$$

and for $\sigma = 0$ is $\qquad {}_{\vec{m}}\mathcal{H}_n(p) = \mathbb{Z}$ for all $n \in \mathbb{N}_0$.

Because of $\partial_n \circ \partial_{n+1} = 0$ we can define ${}_{\vec{m}}\mathcal{H}_n(X) := \frac{kernel(\partial_n)}{image(\partial_{n+1})}$, and this leads to a functor **TOP** $\longrightarrow$ $\mathcal{R}$-**MOD**:
Let $f : X \to Y$ be continuous and $T \in \mathcal{CC}_{n,X}$, then follows $f \circ T \in \mathcal{CC}_{n,Y}$.
Let $\Phi\left[(\mathcal{R})\,\mathcal{CC}\right]_n(f)(T) := f \circ T$.
Then $\Phi\left[(\mathcal{R})\,\mathcal{CC}\right]_n\,(\,X \xrightarrow{f} Y\,) := \Phi\left[(\mathcal{R})\,\mathcal{CC}\right]_n(X) \overset{\Phi[(\mathcal{R})\mathcal{CC}]_n(f)}{\longrightarrow} \Phi\left[(\mathcal{R})\,\mathcal{CC}\right]_n(Y)$
is well defined by linear continuation, hence for an arbitrary $(f : X \to Y) \in$ **TOP** the following diagram commutes in $\mathcal{R}$-**MOD** for all $n \in \mathbb{N}_0$.

$$\begin{array}{ccccccccc}
\ldots & \xrightarrow{\partial_{n+2}} & \Phi_{\mathcal{R},n+1}(X) & \xrightarrow{\partial_{n+1}} & \Phi_{\mathcal{R},n}(X) & \xrightarrow{\partial_n} & \Phi_{\mathcal{R},n-1}(X) & \xrightarrow{\partial_{n-1}} & \ldots \\
\ldots & & \downarrow \Phi_{n+1}(f) & & \downarrow \Phi_n(f) & & \downarrow \Phi_{n-1}(f) & & \ldots \\
\ldots & \xrightarrow{\partial_{n+2}} & \Phi_{\mathcal{R},n+1}(Y) & \xrightarrow{\partial_{n+1}} & \Phi_{\mathcal{R},n}(Y) & \xrightarrow{\partial_n} & \Phi_{\mathcal{R},n-1}(Y) & \xrightarrow{\partial_{n-1}} & \ldots
\end{array}$$

that means for all $n \qquad \Phi_{n-1}(f) \circ \partial_n = \partial_n \circ \Phi_n(f)$.
Thus $\Phi_n(f)$ maps cycles to cycles and boundaries to boundaries, hence for all $(f : X \to Y) \in$ **TOP** one can define
${}_{\vec{m}}\mathcal{H}_n(X \xrightarrow{f} Y) := {}_{\vec{m}}\mathcal{H}_n(X) \overset{{}_{\vec{m}}\mathcal{H}_n(f)}{\longrightarrow} {}_{\vec{m}}\mathcal{H}_n(Y)$, and we get a functor **TOP** $\longrightarrow$ $\mathcal{R}$-**MOD**.
In the same way ${}_{\vec{m}}\mathcal{H}_n$ will be extended to a functor $\mathbf{TOP^2} \longrightarrow \mathcal{R}$-**MOD**:
(The following discription is rather brief; for more details the reader should study [1], [2], [4], [5], or other books about homology theory).
If one has $(f : (X, A) \to (Y, B)) \in \mathbf{TOP^2}$, we have $A \hookrightarrow X$ and $B \hookrightarrow Y$ (in **TOP**) , and also $\Phi\left[(\mathcal{R})\,\mathcal{CC}\right]_n(A) \hookrightarrow \Phi\left[(\mathcal{R})\,\mathcal{CC}\right]_n(X)$ as well as $\Phi\left[(\mathcal{R})\,\mathcal{CC}\right]_n(B) \hookrightarrow \Phi\left[(\mathcal{R})\,\mathcal{CC}\right]_n(Y)$ (in $\mathcal{R}$-**MOD**) .
Hence the $\mathcal{R}$-Moduls $\quad \Phi_{\mathcal{R},n}(X, A) := \Phi\left[(\mathcal{R})\,\mathcal{CC}\right]_n(X, A) := \frac{\Phi[(\mathcal{R})\mathcal{CC}]_n(X)}{\Phi[(\mathcal{R})\mathcal{CC}]_n(A)}$
and $\quad \Phi_{\mathcal{R},n}(Y, B) := \Phi\left[(\mathcal{R})\,\mathcal{CC}\right]_n(Y, B) := \frac{\Phi[(\mathcal{R})\mathcal{CC}]_n(Y)}{\Phi[(\mathcal{R})\mathcal{CC}]_n(B)}$ are well defined.
For $\mathrm{u} \in \Phi\left[(\mathcal{R})\,\mathcal{CC}\right]_n(X)$ let $[\mathrm{u}]_\sim \in \Phi\left[(\mathcal{R})\,\mathcal{CC}\right]_n(X, A)$ be the equivalence class of u modulo $\Phi\left[(\mathcal{R})\,\mathcal{CC}\right]_n(A)$.
Thus $[\mathrm{u}]_\sim = [\mathrm{w}]_\sim$ if and only if $\mathrm{u} - \mathrm{w} \in \Phi\left[(\mathcal{R})\,\mathcal{CC}\right]_n(A)$.
For $(f : (X, A) \to (Y, B)\,) \in \mathbf{TOP^2}$ and $\mathrm{T} \in \mathcal{CC}_{n,A}$ we have $f \circ T \in \mathcal{CC}_{n,B}$ (because of $f(A) \subset B$).
Hence let $\Phi\left[(\mathcal{R})\,\mathcal{CC}\right]_n(f)\,(\,[\mathrm{u}]_\sim\,) := [\,\Phi\left[(\mathcal{R})\,\mathcal{CC}\right]_n(f)\ (\mathrm{u})\,]_\sim \in \Phi\left[(\mathcal{R})\,\mathcal{CC}\right]_n(Y, B)$,
and $\Phi\left[(\mathcal{R})\,\mathcal{CC}\right]_n(f)$ is well defined.
So $\Phi\left[(\mathcal{R})\,\mathcal{CC}\right]_n$ yields a functor : **TOP** $\longrightarrow$ $\mathcal{R}$-**MOD** as well as $\mathbf{TOP^2} \longrightarrow \mathcal{R}$-**MOD** .
As an abbreviation take $\Phi_n(f) := \Phi\left[(\mathcal{R})\,\mathcal{CC}\right]_n(f)$.
And as above, for an arbitrary $(f : X \to Y) \in$ **TOP** the following diagram commutes in $\mathcal{R}$-**MOD** for all $n \in \mathbb{N}_0$.

$$\begin{array}{ccccccccc}
\ldots & \xrightarrow{\partial_{n+2}} & \Phi_{\mathcal{R},n+1}(X,A) & \xrightarrow{\partial_{n+1}} & \Phi_{\mathcal{R},n}(X,A) & \xrightarrow{\partial_n} & \Phi_{\mathcal{R},n-1}(X,A) & \xrightarrow{\partial_{n-1}} & \ldots \\
\ldots & & \downarrow \Phi_{n+1}(f) & & \downarrow \Phi_n(f) & & \downarrow \Phi_{n-1}(f) & & \ldots \\
\ldots & \xrightarrow{\partial_{n+2}} & \Phi_{\mathcal{R},n+1}(Y,B) & \xrightarrow{\partial_{n+1}} & \Phi_{\mathcal{R},n}(Y,B) & \xrightarrow{\partial_n} & \Phi_{\mathcal{R},n-1}(Y,B) & \xrightarrow{\partial_{n-1}} & \ldots
\end{array}$$

If one has $\mathrm{u} \in \Phi_{\mathcal{R},n}(X, A) := \Phi\left[(\mathcal{R})\,\mathcal{CC}\right]_n(X, A)$ with $\partial_n\,(\mathrm{u}) \subset A$, let $[\mathrm{u}]_\sim \in {}_{\vec{m}}\mathcal{H}_n(X, A)$ be the equivalence class modulo image(∂_{n+1}) .
For $(\,f : (X, A) \to (Y, B)\,)$ let ${}_{\vec{m}}\mathcal{H}_n(f)([\,\mathrm{u}\,]_\sim) :\,= [\,\Phi_n(f)(\mathrm{u})]_\sim$ with
$\partial_n\,(\,\Phi_n(f)(\mathrm{u})\,) = \Phi_{n-1}(f) \circ \partial_n\,(\mathrm{u}) \subset B$, hence, as above $\Phi_n(f)$ maps cycles to cycles and boundaries to boundaries, hence $[\,\Phi_n(f)(\mathrm{u})\,]_\sim \in {}_{\vec{m}}\mathcal{H}_n(Y, B)$, and ${}_{\vec{m}}\mathcal{H}_n(f)$ is well defined.
As an abbreviation take $\qquad {}_{\vec{m}}\mathcal{H}_n[\,(X, A) \xrightarrow{f} (Y, B)\,] =:\, {}_{\vec{m}}\mathcal{H}_n(X, A) \xrightarrow{f_*} {}_{\vec{m}}\mathcal{H}_n(Y, B)$.

4 The Homotopy Axiom

Some useful definitions :

Definition 4. *Two continuous functions f , g : $X \longrightarrow Y$ are called homotop $\Longleftrightarrow$ there is a continuous $H : X \times I \longrightarrow Y$ and for all $x \in X$ we have H (x,0) = f(x) and H(x,1) = g(x) . (In brief : $f \approx g$).*

Definition 5. *Let $L \in \mathbb{N}$, and $\vec{m} = [m_0, m_1, m_2, \dots, m_L] \in \mathcal{R}^{L+1}$. ${}_{\vec{m}}\mathcal{H}$ is satisfying the Homotopy Axiom $\Longleftrightarrow$ for all $n \in \mathbb{N}_0$ and for all f , g : $X \longrightarrow Y$ with $f \approx g$ holds that ${}_{\vec{m}}\mathcal{H}_n(f) = {}_{\vec{m}}\mathcal{H}_n(g)$.*

Definition 6. *Let $\mathcal{R}$ be a commutativ ring with unit 1 . Let $\mathcal{M} \subset \mathcal{R}$. $\mathcal{M}$ is called full $\Longleftrightarrow$ the ideal generated by $\mathcal{M}$ is $\mathcal{R}$ $\Longleftrightarrow$ there is a number $n \in \mathbb{N}$ and there are sets $\{ m_1, m_2, \dots , m_n \} \subset \mathcal{M}$ and $\{ r_1, r_2, \dots , r_n \} \subset \mathcal{R}$ such that $\sum_{i=1}^{n} r_i \cdot m_i = 1$, the unit of $\mathcal{R}$.*

Lemma 1. *${}_{\vec{m}}\mathcal{H}$ satisfies the homotopy axiom $\Longleftrightarrow$ for all topological spaces X and for e_0 , e_1 : $X \longrightarrow X \times I$, e_0 (x) := (x,0) and e_1(x) : = (x,1), the equation ${}_{\vec{m}}\mathcal{H}_n(e_0) = {}_{\vec{m}}\mathcal{H}_n(e_1)$ holds for every $n \in \mathbb{N}_0$.*

Proof. "$\Leftarrow$" : If we assume $f \approx g$ there is a H with $f = H \circ e_0$ and $g = H \circ e_1$, and ${}_{\vec{m}}\mathcal{H}_n$ is a functor, hence ${}_{\vec{m}}\mathcal{H}_n(f) = {}_{\vec{m}}\mathcal{H}_n(H) \circ {}_{\vec{m}}\mathcal{H}_n(e_0) = {}_{\vec{m}}\mathcal{H}_n(H) \circ {}_{\vec{m}}\mathcal{H}_n(e_1) = {}_{\vec{m}}\mathcal{H}_n(g)$.
"$\Rightarrow$" : Because of $e_0 = \mathrm{Id}_{X\times I} \circ e_0$ and $e_1 = \mathrm{Id}_{X\times I} \circ e_1$ holds $e_0 \approx e_1$. Qed □

Theorem 2. *HOMOTOPY AXIOM*
Let $L \in \mathbb{N}$ and let $\vec{m} = [m_0, m_1, m_2, \dots, m_L] \in \mathcal{R}^{L+1}$.
We have the equivalence
${}_{\vec{m}}\mathcal{H}$ is satisfying the homotopy axiom $\Longleftrightarrow$ $\{ m_0, m_1, m_2, \dots ,m_L \}$ is full

Proof. '$\Longleftarrow$' : We assume that $\{ m_0, m_1, m_2, \dots, m_L \}$ is full.
Because of the assumption there exists a set $\{ r_0, r_1, \dots, r_L \} \subset \mathcal{R}$ with $\sum_{i=0}^{L} r_i \cdot m_i = 1$.
First we will construct $L+1$ continuous auxiliary functions.
For all $L \in \mathbb{N}$ and for all $k \in \{0, 1, 2, \dots, L\}$ we define a map $\eta_k : [0,1] \rightarrow [0,1]$.
The functions η_k are mostly 0 and have at the place $\frac{k}{L}$ a 'jag' of height 1 .
More precisely :

$$\eta_0(x) := \begin{cases} 1 - Lx & for\ x \in [0, \frac{1}{L}] \\ 0 & for\ x \in [\frac{1}{L}, 1] \end{cases}$$

$$\eta_L(x) := \begin{cases} 0 & for\ x \in [0, \frac{L-1}{L}] \\ Lx - L + 1 & for\ x \in [\frac{L-1}{L}, 1] \end{cases}$$

As well as for $k \in \{1, 2, \dots, L-1 \}$, η_k is the polygon through the five points (0,0), ($\frac{k-1}{L}$,0), ($\frac{k}{L}$,1), ($\frac{k+1}{L}$,0) and (1,0).

$$\eta_k(x) := \begin{cases} 0 & for\ x \in [0, \frac{k-1}{L}] \cup [\frac{k+1}{L}, 1] \\ Lx - k + 1 & for\ x \in [\frac{k-1}{L}, \frac{k}{L}] \\ 1 - Lx + k & for\ x \in [\frac{k}{L}, \frac{k+1}{L}] \end{cases}$$

Note that for $j, k \in \{0, 1, 2, \dots, L\}$ we have $\eta_k(\frac{j}{L}) = \delta_{j,k}$, that means $\eta_k(\frac{j}{L}) = 1$ if $k = j$ and $\eta_k(\frac{j}{L}) = 0$ if $k \neq j$. The maps $e_0, e_1 : X \rightarrow X \times I$ are inducing canonically

two maps $e_0, e_1 : \mathcal{C}C_{n,X} \to \mathcal{C}C_{n,X\times I}$ by $e_i(T) := e_i \circ T$, (for $i \in \{0,1\}$) and by linear continuation two maps $e_0, e_1 : \Phi_{\mathcal{R},n}(X) \to \Phi_{\mathcal{R},n}(X \times I)$.
Now we will define 'chain homotopies', Θ_n: $\Phi_{\mathcal{R},n}(X) \longrightarrow \Phi_{\mathcal{R},n+1}(X \times I)$,
with the property ${}_{\vec{m}}\partial_{n+1} \circ \Theta_n = e_0 - e_1 + \Theta_{n-1} \circ {}_{\vec{m}}\partial_n$.
More precisely: For $n \in \mathbb{N}$ and for $k \in \{0, 1, 2, \ldots, L\}$ define:
ξ_n , $\psi_{n,k}$: $\mathcal{C}C_{n,X} \longrightarrow \mathcal{C}C_{n+1,X\times I}$ as follows:
For every $T : W^n \longrightarrow X$, and for all tuples $(a_1, a_2, \ldots, a_{n+1}) \in W^{n+1}$ let
$\xi_n(T)\ (a_1, a_2, \ldots, a_{n+1}) := (\ T(a_1, a_2, \ldots, a_n), 0\ )$ and
$\psi_{n,k}(T)\ (a_1, a_2, \ldots, a_{n+1}) := (\ T(a_1, a_2, \ldots, a_n), \eta_k(a_{n+1})\ )$,
and for $n = 0$ let $\xi_0(T)\ (a_1) := (\ T(0), 0\ )$ and $\psi_{0,k}(T)\ (a_1) := (\ T(0), \eta_k(a_1)\ )$.
Finally let for all $n \in \mathbb{N}_0$

$$\Theta_n(T) := \sum_{k=0}^{L} r_k \cdot (\ \xi_n(T) - \psi_{n,k}(T)\ ) \tag{6}$$

and $\Theta_{-1} := 0$.
For $\mathrm{u} \in \Phi_{\mathcal{R},n}(X)$ let $\Theta_n(\mathrm{u})$ be defined by linear extension.
Then we have for all $n \in \mathbb{N}_{-1}$ a map $\Theta_n : \Phi_{\mathcal{R},n}(X) \longrightarrow \Phi_{\mathcal{R},n+1}(X \times I)$.
Thus we get the following (noncommutative) diagram in $\mathcal{R}$-**MOD**.

$$\begin{array}{ccccccccc}
\ldots & \xrightarrow{\partial_{n+2}} \Phi_{\mathcal{R},n+1}(X) & \xrightarrow{\partial_{n+1}} & \Phi_{\mathcal{R},n}(X) & \xrightarrow{\partial_n} & \Phi_{\mathcal{R},n-1}(X) & \xrightarrow{\partial_{n-1}} & \ldots \\
\ldots & e_0 \downarrow \downarrow e_1 & \Theta_n \swarrow & e_0 \downarrow \downarrow e_1 & \Theta_{n-1} \swarrow & e_0 \downarrow \downarrow e_1 & & \ldots \\
\ldots & \xrightarrow{\partial_{n+2}} \Phi_{\mathcal{R},n+1}(X \times I) & \xrightarrow{\partial_{n+1}} & \Phi_{\mathcal{R},n}(X \times I) & \xrightarrow{\partial_n} & \Phi_{\mathcal{R},n-1}(X \times I) & \xrightarrow{\partial_{n-1}} & \ldots
\end{array}$$

Now we show that for all $n \in \mathbb{N}_0$ and for all $T \in \mathcal{C}C_{n,X}$ holds that

$$[{}_{\vec{m}}\partial_{n+1} \circ \Theta_n]\,(T) = [(-1)^{n+2}(e_0 - e_1) + \Theta_{n-1} \circ\ {}_{\vec{m}}\partial_n](T). \tag{7}$$

For $n = 0$ the proof is a more simple version of the following one and will be omitted. For $n \in \mathbb{N}$ one has: $[{}_{\vec{m}}\partial_{n+1} \ \circ\ \Theta_n]\,(T) \ = \ {}_{\vec{m}}\partial_{n+1} \left[\ \sum_{k=0}^{L} r_k\ (\xi_n(T) - \psi_{n,k}(T)\ )\ \right]$

$$\begin{aligned}
&= \sum_{k=0}^{L} r_k \sum_{j=1}^{n+1} (-1)^{j+1} \sum_{i=0}^{L} m_i \left[\langle \xi_n(T)\rangle_{n+1,i,j} - \langle \psi_{n,k}(T)\rangle_{n+1,i,j}\right] \\
&= \sum_{j=1}^{n} (-1)^{j+1} \sum_{i,k=0}^{L} r_k\ m_i \left[\langle \xi_n(T)\rangle_{n+1,i,j} - \langle \psi_{n,k}(T)\rangle_{n+1,i,j}\right] \\
&\quad + (-1)^{n+2} \sum_{i,k=0}^{L} r_k\ m_i \left[\langle \xi_n(T)\rangle_{n+1,i,n+1} - \langle \psi_{n,k}(T)\rangle_{n+1,i,n+1}\right] \\
&= \quad \mathrm{Rest}\ +\ \mathrm{D}
\end{aligned}$$

with $\quad \mathrm{Rest} := \displaystyle\sum_{j=1}^{n} (-1)^{j+1} \sum_{i,k=0}^{L} r_k\ m_i \left[\langle \xi_n(T)\rangle_{n+1,i,j} - \langle \psi_{n,k}(T)\rangle_{n+1,i,j}\right]$

and $\quad \mathrm{D} := (-1)^{n+2} \displaystyle\sum_{i,k=0}^{L} r_k\ m_i \left[\langle \xi_n(T)\rangle_{n+1,i,n+1} - \langle \psi_{n,k}(T)\rangle_{n+1,i,n+1}\right].$

We have for all $(a_1, a_2, \dots, a_n) \in W^n$:

$$\begin{aligned}
& \Big[\langle \xi_n(T) \rangle_{n+1,i,n+1} - \langle \psi_{n,k}(T) \rangle_{n+1,i,n+1} \Big] \quad (a_1, a_2, \dots, a_n) \\
= \; & \xi_n(T)(a_1, a_2, \dots, a_n, \frac{i}{L}) - \psi_{n,k}(T)(a_1, a_2, \dots, a_n, \frac{i}{L}) \\
= \; & (T(a_1, a_2, \dots, a_n), 0) - (T(a_1, a_2, \dots, a_n), \eta_k(\frac{i}{L}))
\end{aligned}$$

Since $\eta_k(\frac{i}{L}) = \delta_{i,k}$ and $\sum_{k=0}^{L} r_k \; m_k = 1$ it follows that

$$\begin{aligned}
D &= (-1)^{n+2} \sum_{k=0}^{L} r_k \; m_k \Big[\langle \xi_n(T) \rangle_{n+1,k,n+1} - \langle \psi_{n,k}(T) \rangle_{n+1,k,n+1} \Big] \\
&= (-1)^{n+2} \sum_{k=0}^{L} r_k \; m_k \, [e_0 \circ T - e_1 \circ T] \\
&= (-1)^{n+2} (e_0 \circ T - e_1 \circ T) \sum_{k=0}^{L} r_k \; m_k \\
&= (-1)^{n+2} (e_0 \circ T - e_1 \circ T) \\
&= (-1)^{n+2} (e_0(T) - e_1(T))
\end{aligned}$$

It remains to show that Rest $= [\, \Theta_{n-1} \circ {}_{\vec{m}}\partial_n \,] \, (T)$. One has

$$\begin{aligned}
[\, \Theta_{n-1} \circ {}_{\vec{m}}\partial_n \,] \, (T) &= \Theta_{n-1} \; \Big(\sum_{j=1}^{n} (-1)^{j+1} \sum_{i=0}^{L} m_i \; \langle T \rangle_{\, n, \, i, \, j} \Big) \\
&= \sum_{j=1}^{n} (-1)^{j+1} \sum_{i=0}^{L} m_i \; \Theta_{n-1} \; (\; \langle T \rangle_{\, n, \, i, \, j} \;) \\
&= \sum_{j=1}^{n} (-1)^{j+1} \sum_{i=0}^{L} m_i \; \sum_{k=0}^{L} r_k \; \; (\; \xi_{n-1}(\; \langle T \rangle_{\, n, \, i, \, j} \;) - \psi_{n-1,k}(\; \langle T \rangle_{\, n, \, i, \, j} \;) \;) \\
&= \sum_{j=1}^{n} (-1)^{j+1} \sum_{i,k=0}^{L} m_i \; r_k \; \; (\; \xi_{n-1}(\; \langle T \rangle_{\, n, \, i, \, j} \;) - \psi_{n-1,k}(\; \langle T \rangle_{\, n, \, i, \, j} \;) \;)
\end{aligned}$$

We have for all $n \in \mathbb{N} \backslash \{1\}$, for $T \in \mathcal{C}C_{n,X}$, for $j \in \{1, \, 2, \, \dots, n\}$, and for $i, k \in \{0, \, 1, \, 2, \, \dots \, , \, L\}$ for every n-tuple $(a_1, a_2, \dots, a_n) \in W^n$:

$$\begin{aligned}
\xi_{n-1}(\; \langle T \rangle_{\, n, \, i, \, j} \;)(a_1, a_2, \dots, a_n) &= (\langle T \rangle_{\, n, \, i, \, j} \; (a_1, a_2, \dots, a_{n-1}), 0) \\
&= (T(a_1, a_2, \dots, a_{j-1}, \frac{i}{L}, a_j, \dots, a_{n-1}), 0) \\
&= \xi_n(T)(a_1, a_2, \dots, a_{j-1}, \frac{i}{L}, a_j, \dots, a_n) \\
&= \langle \xi_n(T) \rangle_{n+1,i,j} \, (a_1, a_2, \dots, a_n).
\end{aligned}$$

In short, we get : $\xi_{n-1}(\; \langle T \rangle_{\, n, \, i, \, j} \;) = \langle \xi_n(T) \rangle_{n+1,i,j}$.

In the same way we find that

$$\begin{aligned}
\psi_{n-1,k}(\ \langle T\rangle_{\ n,\ i,\ j}\ )(a_1,a_2,\ldots,a_n) &= (\langle T\rangle_{\ n,\ i,\ j}\ (a_1,a_2,\ldots,a_{n-1}),\eta_k(a_n)) \\
&= (T(a_1,a_2,\ldots,a_{j-1},\frac{i}{L},a_j,\ldots,a_{n-1}),\eta_k(a_n)) \\
&= \psi_{n,k}(T)(a_1,a_2,\ldots,a_{j-1},\frac{i}{L},a_j,\ldots,a_n) \\
&= \langle\psi_{n,k}(T)\rangle_{n+1,i,j}\,(a_1,a_2,\ldots,a_n),
\end{aligned}$$

therefore $\psi_{n-1,k}(\ \langle T\rangle_{\ n,\ i,\ j}\ ) = \langle\psi_{n,k}(T)\rangle_{n+1,i,j}$,
and finally $[\ \Theta_{n-1}\ \circ\ {}_{\vec{m}}\partial_n\ ]\ (T) = \text{Rest}$ follows.

Hence we have proved that ${}_{\vec{m}}\partial_{n+1}\circ\Theta_n = (-1)^{n+2}\ (\ e_0\ -\ e_1\ ) + \Theta_{n-1}\circ{}_{\vec{m}}\partial_n$.
In particular, for a cycle $\mathrm{u}\in\Phi_{\mathcal{R},n}(X)$ (that means ${}_{\vec{m}}\partial_n\ (\mathrm{u}) = 0$) holds that $[{}_{\vec{m}}\partial_{n+1}\circ\Theta_n]\ (\mathrm{u}) = [\ (-1)^{n+2}\ (\ e_0\ -\ e_1)]\ (\mathrm{u})\ \ \in\ \text{image}({}_{\vec{m}}\partial_{n+1})$,
hence one can deduce for the equivalence class $[\mathrm{u}]_\sim\in{}_{\vec{m}}\mathcal{H}_n(\mathrm{X})$ that
${}_{\vec{m}}\mathcal{H}_n(e_0)\ (\ [\mathrm{u}]_\sim\ )\ -\ {}_{\vec{m}}\mathcal{H}_n(e_1)\ (\ [\mathrm{u}]_\sim) = 0$, and therefore ${}_{\vec{m}}\mathcal{H}_n(e_0) = {}_{\vec{m}}\mathcal{H}_n(e_1)$.
By Lemma 1 the homotopy axiom is satisfied.

'$\Longrightarrow$' : We assume that $\{\ m_0,m_1,m_2,\ldots,m_L\ \}$ is not full, that means that for all $(L+1)$-subsets $\{\ a_0,\ a_1,\ \ldots\ ,\ a_L\ \}\subset\mathcal{R}$ holds that $\sum_{i=0}^{L} a_i\,m_i \neq 1$.
Take a space X which should be sufficient complex, for instance $X := W^2$, and for $k\in\mathbb{N}$ let T_1 , $T_2 : W^k\to W^2$ with $T_1\neq T_2$, but ${}_{\vec{m}}\partial_k\ (\ T_1\ ) = {}_{\vec{m}}\partial_k\ (\ T_2\ )$.
Let $\mathrm{u} := T_1 - T_2\in\Phi_{\mathcal{R},k}(X)$.
Because of ${}_{\vec{m}}\partial_k\ (\ \mathrm{u}\ ) = 0$, u is a cycle, i. e. $[\ \mathrm{u}\ ]_\sim\in\mathcal{H}_k(\mathrm{X})$.
For $i\ \in\{0{,}1\}$ let $\mathrm{e}_{k,i} : \Phi_{\mathcal{R},k}(X)\longrightarrow\Phi_{\mathcal{R},k}(X\times I)$, $\mathrm{e}_{k,i}(T)\ (\vec{a}) := (T(\vec{a}),i)$ as before.
The boundary operator ${}_{\vec{m}}\partial_k$ is a natural transformation, hence for $i\in\{0{,}1\}$ the following diagram commutes:

$$\begin{array}{ccccc}
 & & \Phi_{\mathcal{R},k}(X) & \xrightarrow{\partial_k} & \Phi_{\mathcal{R},k-1}(X) \\
 & & e_{k,i}\downarrow & & e_{k-1,i}\downarrow \\
\Phi_{\mathcal{R},k+1}(X\times I) & \xrightarrow{\partial_{k+1}} & \Phi_{\mathcal{R},k}(X\times I) & \xrightarrow{\partial_k} & \Phi_{\mathcal{R},k-1}(X\times I)
\end{array}$$

Because of ${}_{\vec{m}}\partial_k\ (\ \mathrm{u}\ ) = 0$, one has $[\ e_{k-1,i}\circ{}_{\vec{m}}\partial_k\ ]\ (\ \mathrm{u}\ ) = 0 = [\ {}_{\vec{m}}\partial_k\circ e_{k,i}\ ]\ (\ \mathrm{u}\ )$, and we get that $e_{k,i}(\ \mathrm{u}\ )$ is a cycle in $X\times I$, thus $[\ e_{k,i}(\ \mathrm{u}\ )\ ]_\sim\in\mathcal{H}_k(\mathrm{X}\times I)$.
Now we want to show that $[\ e_{k,0}(\mathrm{u})\ ]_\sim\neq[\ e_{k,1}(\mathrm{u})\ ]_\sim$.
We assume the equality, thus
$[\ e_{k,0}(\ \mathrm{u}\ )\ ]_\sim = [\ e_{k,1}(\ \mathrm{u}\ )\ ]_\sim \Longleftrightarrow [\ e_{k,0}(\mathrm{u}) - e_{k,1}(\mathrm{u})\ \ ]_\sim = 0\in\mathcal{H}_k(X\times I) \Longleftrightarrow$
$e_{k,0}(\ \mathrm{u}\ ) - e_{k,1}(\ \mathrm{u}\ )$ is a boundary $\Longleftrightarrow$
there is a $\psi\in\Phi_{\mathcal{R},k+1}(X\times I)$ and $e_{k,0}(\ \mathrm{u}\ ) - e_{k,1}(\ \mathrm{u}\ ) = {}_{\vec{m}}\partial_{k+1}(\psi)$.
An element $\psi\ \in\Phi_{\mathcal{R},k+1}(X\times I)$ means, that there are elements $p\in\mathbb{N}$, $r_1,r_2,\ldots,r_p\in\mathcal{R}$ and $\psi_1\ ,\ \psi_2\ ,\ \ldots\ ,\ \psi_p\in\mathcal{CC}_{k+1,X\times I}$ such that $\psi = \sum_{t=1}^{p} r_t\cdot\psi_t$.
Now we define four functions $T_{1,0},T_{2,0},T_{1,1},T_{2,1}\in\mathcal{CC}_{k,X\times I}$ with the proporties
$e_{k,0}(\ \mathrm{u}\ ) - e_{k,1}(\ \mathrm{u}\ ) = T_{1,0} - T_{2,0} - T_{1,1} + T_{2,1}$. For all $\vec{a}\in\mathrm{W}^k$ let
$T_{1,0}(\vec{a}) := (\ T_1(\vec{a}),0\ )$, $T_{2,0}(\vec{a}) := (\ T_2(\vec{a}),0\ )$, $T_{1,1}(\vec{a}) := (\ T_1(\vec{a}),1\ )$, $T_{2,1}(\vec{a}) := (\ T_2(\vec{a}),1\ )$,
that means $T_{1,0} = e_{k,0}\circ T_1$, $T_{2,0} = e_{k,0}\circ T_2$, $T_{1,1} = e_{k,1}\circ T_1$ and $T_{2,1} = e_{k,1}\circ T_2$.
Because of $T_1\neq T_2$ the four maps $T_{1,0},T_{2,0},T_{1,1}$ and $T_{2,1}$ are different two by two.
Hence it holds that

$T_{1,0} - T_{2,0} - T_{1,1} + T_{2,1} \;=\; e_{k,0}(T_1 - T_2) - e_{k,1}(T_1 - T_2) \;=\; e_{k,0}(\text{ u }) - e_{k,1}(\text{ u }) \;=\; {}_{\vec{m}}\partial_{k+1}(\psi)$
$= \;\sum_{t=1}^{p} r_t \; {}_{\vec{m}}\partial_{k+1}(\psi_t) \;=\; \sum_{t=1}^{p} r_t \sum_{j=1}^{k+1} (-1)^{j+1} \sum_{i=0}^{L} m_i \;\langle \psi_t \rangle_{\,k+1,\,i,\,j}$, thus
$\sum_{t=1}^{p} r_t \sum_{j=1}^{k+1} (-1)^{j+1} \sum_{i=0}^{L} m_i \;\langle \psi_t \rangle_{\,k+1,\,i,\,j} \;=\; 1 \cdot T_{1,0} - 1 \cdot T_{2,0} - 1 \cdot T_{1,1} + 1 \cdot T_{2,1}$.

On the left hand side there are $p(k+1)(L+1)$ elements out of $\mathcal{CC}_{k,X\times I}$ which generate the four elemets on the right hand side. Let :
$B := \{ \;(t,j,i\;) \mid t \in \{1,2,...,p\}\;,\, j \in \{1,2,...,k+1\},\, i \in \{0,1,...,L\} \;\mid \langle \psi_t \rangle_{\,k+1,\,i,\,j} \;= T_{1,0}\; \}$.
Then we have :

$$\sum_{(t,j,i)\in B} r_t \;(-1)^{j+1}\; m_i \;\cdot \langle \psi_t \rangle_{\,k+1,\,i,\,j} \;=\; \sum_{(t,j,i)\in B} r_t \;(-1)^{j+1}\; m_i \;\cdot\; T_{1,0} \;=\; 1 \cdot T_{1,0} \qquad , \qquad (8)$$

a contradiction to the assumption. This finishes the proof. □

5 The Exact Sequence of a Pair

As we mentioned above, for all $n \in \mathbb{N}_0$ the boundary operator yields a functor
${}_{\vec{m}}\mathcal{H}_n : \mathbf{TOP^2} \longrightarrow \mathcal{R}\text{-}\mathbf{MOD}$,
that means for any $f : ((X,A) \longrightarrow (Y,B))$ we have ${}_{\vec{m}}\mathcal{H}_n(f) : {}_{\vec{m}}\mathcal{H}_n(X,A) \longrightarrow {}_{\vec{m}}\mathcal{H}_n(Y,B)$,
and for any $\text{u} \in \Phi\left[(\mathcal{R})\,\mathcal{CC}\right]_n$ with ${}_{\vec{m}}\partial_n(u)$ is a chain in A, (hence $[\text{u}]_\sim \in {}_{\vec{m}}\mathcal{H}_n(X,A)$),
holds : ${}_{\vec{m}}\mathcal{H}_n(f)\;(\;[\text{u}]_\sim) = [\;{}_{\vec{m}}\mathcal{H}_n\;(f)(u)]_\sim$.
As an abreviation we define for all topological spaces X : $\Phi_{\mathcal{R},n}(X) \;:= \Phi\left[(\mathcal{R})\,\mathcal{CC}\right]_n(X)$.
For a subspace $A \subset X$ one gets a short exact sequence of $\mathcal{R}$-Moduls :
$\{\;0\;\} \longrightarrow \Phi_{\mathcal{R},n}(A) \longrightarrow \Phi_{\mathcal{R},n}(X) \longrightarrow \Phi_{\mathcal{R},n}(X,A) \longrightarrow \{\;0\;\}$.
Together with the boundary operators $(\;{}_{\vec{m}}\partial_n)_{n\in\mathbb{N}_0}$ one gets a short exact sequence of chain-complexes:
$\{\;0\;\} \longrightarrow {}_{\vec{m}}\mathcal{K}(A) \longrightarrow {}_{\vec{m}}\mathcal{K}(X) \longrightarrow {}_{\vec{m}}\mathcal{K}(X,A) \longrightarrow \{\;0\;\}$.
Let i: $A \hookrightarrow X$ and $j\;(X,\emptyset) \hookrightarrow (X,A)$ be the topological inclusions.
Now one can construct for all $n \in \mathbb{N}_0$ a morphism:
$\text{k}_* : {}_{\vec{m}}\mathcal{H}_n(X,A) \longrightarrow {}_{\vec{m}}\mathcal{H}_{n-1}(A)$ of $\mathcal{R}$-modules,
which is called the ' connecting homomorphism '.
This finally yields a long exact sequence of $\mathcal{R}$-module homomorphisms :
...... $\xrightarrow{j_*} {}_{\vec{m}}\mathcal{H}_{n+1}(X,A) \xrightarrow{k_*} {}_{\vec{m}}\mathcal{H}_n(A) \xrightarrow{i_*} {}_{\vec{m}}\mathcal{H}_n(X) \xrightarrow{j_*} {}_{\vec{m}}\mathcal{H}_n(X,A) \xrightarrow{k_*} {}_{\vec{m}}\mathcal{H}_{n-1}(A) \xrightarrow{i_*}$ ......

For details see any book about homology theory , for instance [2], [1], [4], [5] , but it is not necessary for us to repeat all these well known matters of facts.

6 The Excision Axiom

Now we will talk about the so-called ' Excision Axiom '. This is another important theorem in classical singular homology theory, because it helps to compute many homology groups of certain spaces. For a better understanding of our calculations it would be very useful to know the similar method which Prof. Massey presents in [1],26 - 37 ; this knowledge will make the following considerations much easier.
For a topological space X and a subset A let $Int(A)$ be the interior of A, and $Cl(A)$ be the closure of A. To prove the following theorem , we need an extra assumption , which we formulate now.

Definition 7. *Let* $\{\;a,b\;\} \;\subset \mathcal{R}$ *for a commutative Ring* $\mathcal{R}$ *with 1.*
The set $\{a,b\}$ *have the condition* $\mathcal{NCD} \iff$
for all $n \in \mathbb{N}$ *there exist* $x_n\;,\,y_n \in \mathcal{R}$ *with* $x_n \cdot a^n + y_n \cdot b^n = 1$.

Remarks: Of course, $\mathcal{NCD}$ is equivalent to: for all $n \in \mathbb{N}$ the set $\{ a^n , b^n \}$ is full.
$\mathcal{NCD}$ reminds us of 'No Common Divisor'.
In the ring $\mathbb{Z}$ (or in a principal domain $\mathcal{R}$ respectively), holds:
$\{ a, b \}$ has $\mathcal{NCD} \iff \gcd \{a, b\} = 1 \iff \mathrm{Ideal}(\{a, b\}) = \mathcal{R} \iff \{a, b\}$ is full.

Theorem 3. *EXCISION AXIOM*
Let X be a topological space and B and A be subspaces of X.
Let $Cl(B) \hookrightarrow Int(A) \hookrightarrow X$.
Let $i : (X \backslash B, A \backslash B) \hookrightarrow (X, A)$ be the inclusion.
Let $L := 1$, let $\vec{m} := [m_0, m_1] := [a, b] \in \mathcal{R}^2$, let $\{a, b\}$ have the condition $\mathcal{NCD}$.
Then for each $n \in \mathbb{N}_0$ one has isomorphisms:
$i_* : {}_{\vec{m}}\mathcal{H}_n(X \backslash B, A \backslash B) \xrightarrow{\simeq} {}_{\vec{m}}\mathcal{H}_n(X, A)$, which are induced by the inclusion i.

This theorem follows directly from a hard-to-prove proposition, which first needs some technical definitions:
Let I be a set of indices, let $\mathcal{U}$ be a set of subsets of X,
that means $\mathcal{U} := \{U_i \mid \text{for all } i \in I : U_i \subset X\} \subset \{U \mid U \subset X\}$.
Let $\bigcup \{ Int(U_i) \mid i \in I \} = X$.
We call $\mathcal{U}$ a 'generalized open covering of X'. (The U_i need not to be open).
Let $CC_{n,X,\mathcal{U}} := \{ T : W^n \to X \mid T$ is continuous and there is $i \in I$ and $T(W^n) \subset U_i \}$.
Let for all $n \in \mathbb{N}_0$ for every topological spaces X:
$\Phi_{\mathcal{R},n}(X, \mathcal{U})$:= the free $\mathcal{R}$-Modul with the Base $\mathcal{CC}_{n,X,\mathcal{U}}$.
The elements $u \in \Phi_{\mathcal{R},n}(X, \mathcal{U})$ are called: '$\mathcal{U}$ - small chains' .

For a topological space X and $A \subset X$, i. e. $A \overset{i}{\hookrightarrow} X$, the notation $\Phi_{\mathcal{R},n}(A, \mathcal{U})$ means the free $\mathcal{R}$-Modul with base $\mathcal{CC}_{n,A,\mathcal{U}}$.

The inclusion i leads to an inclusion $\widehat{i}$: $\Phi_{\mathcal{R},n}(A, \mathcal{U}) \overset{\widehat{i}}{\hookrightarrow} \Phi_{\mathcal{R},n}(X, \mathcal{U})$.
Define : $\Phi_{\mathcal{R},n}(X, A, \mathcal{U}) := \frac{\Phi_{\mathcal{R},n}(X, \mathcal{U})}{\Phi_{\mathcal{R},n}(A, \mathcal{U})}$.

We have the inclusion $\Phi_{\mathcal{R},n}(X, A, \mathcal{U}) \overset{j}{\hookrightarrow} \Phi_{\mathcal{R},n}(X, A)$.
The boundary operator ${}_{\vec{m}}\partial_n$ commutes with the inclusion $\widehat{i}$, that means for all $T \in \mathcal{CC}_{n,A,\mathcal{U}}$ holds that ${}_{\vec{m}}\partial_n \circ \widehat{i}\ (\mathrm{T}) = \widehat{i} \circ {}_{\vec{m}}\partial_n\ (\mathrm{T})$.
Therefore ${}_{\vec{m}}\partial_n$ induces a map $\Phi_{\mathcal{R},n}(X, A, \mathcal{U}) \longrightarrow \Phi_{\mathcal{R},n-1}(X, A, \mathcal{U})$, which we call ${}_{\vec{m}}\partial_n$, too.
Because of ${}_{\vec{m}}\partial_n \circ {}_{\vec{m}}\partial_{n+1} = 0$ it leads to
'$\mathcal{U}$-small homology $\mathcal{R}$-modules' : ${}_{\vec{m}}\mathcal{H}_n(X, A, \mathcal{U}) := \frac{kernel({}_{\vec{m}}\partial_n)}{image({}_{\vec{m}}\partial_{n+1})}$.
For more details, see [1], pages 29,30 .
Now we are able to prove the following proposition :

Proposition 1. *Let X be topological space and $A \overset{i}{\hookrightarrow} X$ and $\mathcal{U}$ be a generalized open covering of X with the inclusion $\Phi_{\mathcal{R},n}(X, A, \mathcal{U}) \overset{j}{\hookrightarrow} \Phi_{\mathcal{R},n}(X, A)$.*
Let $\vec{m} := [a, b]$ where $\{ a, b \}$ fulfills the condition $\mathcal{NCD}$. Then for all $n \in \mathbb{N}_0$ there is an isomorphism $j_* : {}_{\vec{m}}\mathcal{H}_n(X, A, \mathcal{U}) \xrightarrow{\simeq} {}_{\vec{m}}\mathcal{H}_n(X, A)$, induced by j .

The proof is rather lengthy and will take the next few pages.
With this proposition the excision axiom easily follows, see [1], 30,31 .

It remains the prove of the proposition.
First we'll present the proof with A := $\emptyset$. Afterwards the general case with A $\neq \emptyset$ is an easy application of the Five-Lemma. So let A := $\emptyset$.
We have to define for all $n \in \mathbb{N}_0$ 'subdivision maps', $\mathcal{SD}_n : \Phi_{\mathcal{R},n}(X) \longrightarrow \Phi_{\mathcal{R},n}(X)$.
We need some preparations.
For a $T \in \mathcal{CC}_{n,X}$ we want to 'subdivide' the map T into a few smaller ones. Let :

$q_n : \mathbb{R}^n \longrightarrow W^n$
$(x_1, x_2, \ldots, x_n) \mapsto (y_1, y_2, \ldots, y_n)$ with: For all $i \in \{1, 2, \ldots, n\}$ let:

$$y_i := \begin{cases} 0 & \Leftrightarrow \quad x_i \leq 0 \\ x_i & \Leftrightarrow \quad x_i \in [0,1] \\ 1 & \Leftrightarrow \quad x_i \geq 1 \end{cases}$$

Define for $n \in \mathbb{N}$, $\alpha \in \mathbb{R}$, and for fixed tuples $\vec{v} = (v_1, v_2, \ldots, v_n)$, $\vec{e} = (e_1, e_2, \ldots, e_n) \in \mathbb{R}^n$ for every $T : W^n \longrightarrow X$ the maps:

$\|T\|_{\alpha,\vec{e},\vec{v}} : \quad W^n \to X$ as well as $H_{\alpha,\vec{e},\vec{v}} : \quad W^n \to \mathbb{R}^n$:
For all $(x_1, x_2, \ldots, x_n) \in W^n$ let $H_{\alpha,\vec{e},\vec{v}}(x_1, x_2, \ldots, x_n) := (z_1, z_2, \ldots, z_n)$ with for all $i \in \{1, 2, \ldots, n\}$ let $z_i := \alpha \cdot (e_i + v_i \cdot x_i)$.
And finally define $\|T\|_{\alpha,\vec{e},\vec{v}}\,(x_1, x_2, \ldots, x_n) := (T \circ q_n \circ H_{\alpha,\vec{e},\vec{v}})\,(x_1, x_2, \ldots, x_n)$.

In this paper α will be $\frac{1}{3}$, the $v_i's$ will be ± 1 , and the $e_i's$ will be out of $\{0, 2\}$.
Definition of the $\mathcal{SD}_n$'s :

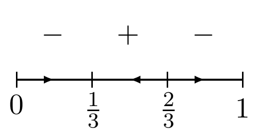
For $n = -1$ let $\mathcal{SD}_{-1}$ be the 0-map; for $n = 0$ let $\mathcal{SD}_0$: $\Phi_{\mathcal{R},0}(\mathrm{X}) \longrightarrow \Phi_{\mathcal{R},0}(X)$,
for all $T : W^0 \to X$ let $\mathcal{SD}_0(T) := -T$.
Define the sets: $\mathcal{E} := \{0, 2\}$ and $\mathcal{V} := \{-1, +1\}$.
Let for all $n \in \mathbb{N}$ and for any $T : W^n \to X$:

$$\mathcal{SD}_n(T) := \sum_{\vec{e} \in \mathcal{E}^n} \sum_{\vec{v} \in \widehat{\mathcal{V}}_n} \left(-\prod_{i=1}^{n} v_i\right) \cdot \|T\|_{\frac{1}{3},\vec{e},\vec{v}} \tag{9}$$

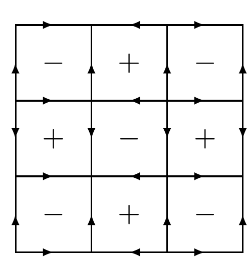
with $\widehat{\mathcal{V}}_n := \{\ (v_1\,,\ v_2\,,\ \ldots\,,\ v_n) \in \mathcal{V}^n \mid$
for all $i = 1, 2, \cdots, n$: $\quad (\ e_i = 0 \Rightarrow v_i = 1) \quad$ and $\quad (e_i = 2 \Rightarrow v_i \in \{-1, 1\}\ )\ \}$
and we get a map $\mathcal{SD}_n : \quad \Phi_{\mathcal{R},n}(X) \longrightarrow \Phi_{\mathcal{R},n}(X)$ by linear extension.
Examples :
For $T \in \mathcal{CC}_{1,X}$, i.e. T: $W^1 \to X$ we have
$\mathcal{SD}_1\ (T) = -\|T\|_{\frac{1}{3},0,1} + \|T\|_{\frac{1}{3},2,-1} - \|T\|_{\frac{1}{3},2,1}$ (see the picture) ,
and for a $T \in \mathcal{CC}_{2,X}$ we get the linear combination : (See the picture, too)

$$\begin{aligned}\mathcal{SD}_2(T) = &-\|T\|_{\frac{1}{3},\binom{0}{0},\binom{1}{1}} - \|T\|_{\frac{1}{3},\binom{2}{0},\binom{1}{1}} + \|T\|_{\frac{1}{3},\binom{2}{0},\binom{-1}{1}} - \|T\|_{\frac{1}{3},\binom{0}{2},\binom{1}{1}} + \|T\|_{\frac{1}{3},\binom{0}{2},\binom{1}{-1}} \\ &+ \|T\|_{\frac{1}{3},\binom{2}{2},\binom{-1}{1}} - \|T\|_{\frac{1}{3},\binom{2}{2},\binom{1}{1}} + \|T\|_{\frac{1}{3},\binom{2}{2},\binom{1}{-1}} - \|T\|_{\frac{1}{3},\binom{2}{2},\binom{-1}{-1}}\end{aligned}$$

$\mathcal{SD}_1(T)$



$\mathcal{SD}_2(T)$



Generally for all $n \in \mathbb{N}_0$ for $T \in \mathcal{CC}_{n,X}$, $\mathcal{SD}_n(T)$ is a linear combination of 3^n maps out of $\mathcal{CC}_{n,X}$.

Lemma 2. *For all $n \in \mathbb{N}_0$ the map $\mathcal{SD}_n$ commutes with the boudary operator ${}_{\vec{m}}\partial_n$, i.e. the following diagram commutes :*

$$\begin{array}{ccccccccc}
\cdots \overset{{}_{\vec{m}}\partial_{n+2}}{\longrightarrow} & \Phi_{n+1}(X) & \overset{{}_{\vec{m}}\partial_{n+1}}{\longrightarrow} & \Phi_n(X) & \overset{{}_{\vec{m}}\partial_n}{\longrightarrow} & \Phi_{n-1}(X) & \overset{{}_{\vec{m}}\partial_{n-1}}{\longrightarrow} & \cdots \\
\cdots & \downarrow \mathcal{SD}_{n+1} & & \downarrow \mathcal{SD}_n & & \downarrow \mathcal{SD}_{n-1} & & \cdots \\
\cdots \overset{{}_{\vec{m}}\partial_{n+2}}{\longrightarrow} & \Phi_{n+1}(X) & \overset{{}_{\vec{m}}\partial_{n+1}}{\longrightarrow} & \Phi_n(X) & \overset{{}_{\vec{m}}\partial_n}{\longrightarrow} & \Phi_{n-1}(X) & \overset{{}_{\vec{m}}\partial_{n-1}}{\longrightarrow} & \cdots
\end{array}$$

Proof. We have to prove that:
${}_{\vec{m}}\partial_n \circ \mathcal{SD}_n(T) = \mathcal{SD}_{n-1} \circ {}_{\vec{m}}\partial_n(T)$ for every $n \in \mathbb{N}_0$ and $T \in \mathcal{CC}_{n,X}$.
This is trivial for $n = 0$ and easy for $n = 1$, so let $n \geq 2$. Let $T \in \mathcal{CC}_{n,X}$.
Note that in the following we will use instead of the expression '$\langle T \rangle_{n,\, i,\, j}$' the expression '$\langle T \rangle_{i,\, j}$' for avoiding the dimension n, to make it better readable. We have :

$$\begin{aligned}
{}_{\vec{m}}\partial_n \circ \mathcal{SD}_n(T) &= {}_{\vec{m}}\partial_n \left[\sum_{\vec{e}\in\mathcal{E}^n} \sum_{\vec{v}\in\widehat{\mathcal{V}}_n} (-\prod_{i=1}^{n} v_i) \cdot \|T\|_{\frac{1}{3},\vec{e},\vec{v}} \right] \\
&= \sum_{j=1}^{n} (-1)^{j+1} \sum_{\vec{e}\in\mathcal{E}^n} \sum_{\vec{v}\in\widehat{\mathcal{V}}_n} (-\prod_{i=1}^{n} v_i) \cdot \left[a \cdot \left\langle \|T\|_{\frac{1}{3},\vec{e},\vec{v}} \right\rangle_{0,j} + b \cdot \left\langle \|T\|_{\frac{1}{3},\vec{e},\vec{v}} \right\rangle_{1,j} \right]
\end{aligned}$$

As well as

$$\begin{aligned}
\mathcal{SD}_{n-1} \circ {}_{\vec{m}}\partial_n(T) &= \mathcal{SD}_{n-1} \left[\sum_{j=1}^{n} (-1)^{j+1} \left(a \cdot \langle T \rangle_{0,j} + b \cdot \langle T \rangle_{1,j} \right) \right] \\
&= \sum_{j=1}^{n} (-1)^{j+1} \sum_{\vec{e}\in\mathcal{E}^{n-1}} \sum_{\vec{v}\in\widehat{\mathcal{V}}_{n-1}} (-\prod_{i=1}^{n-1} v_i) \left[a \cdot \left\| \langle T \rangle_{0,j} \right\|_{\frac{1}{3},\vec{e},\vec{v}} + b \cdot \left\| \langle T \rangle_{1,j} \right\|_{\frac{1}{3},\vec{e},\vec{v}} \right]
\end{aligned}$$

The equality is not obvious; so we have to calculate. It seems that the first sum is 'bigger'. But a lot of pairs of elements will eliminate themselves, and the rest is equal to the second sum.

Let for all $j \in \{1, 2, \cdots, n\}$: $\vec{e} := (e_1,\, e_2,\, \cdots,\, e_{j-1},\, 2,\, e_{j+1},\, \cdots,\, e_n) \in \mathcal{E}^n$,
$\vec{\vartheta_1} := (v_1,\, v_2,\, \cdots,\, v_{j-1},\, 1,\, v_{j+1},\, \cdots,\, v_n) \in \widehat{\mathcal{V}}_n$,
$\vec{\vartheta_{-1}} := (v_1,\, v_2,\, \cdots,\, v_{j-1},\, -1,\, v_{j+1},\, \cdots,\, v_n) \in \widehat{\mathcal{V}}_n$.
We have $\left\langle \|T\|_{\frac{1}{3},\vec{e},\vec{\vartheta_1}} \right\rangle_{0,j}$, $\left\langle \|T\|_{\frac{1}{3},\vec{e},\vec{\vartheta_{-1}}} \right\rangle_{0,j} \in \mathcal{CC}_{n-1,X}$.
Thus, for an element $(x_1,\, x_2,\, \cdots,\, x_{n-1})$ out of W^{n-1} holds:
$\left\langle \|T\|_{\frac{1}{3},\vec{e},\vec{\vartheta_1}} \right\rangle_{0,j} (x_1,\, x_2,\, \cdots,\, x_{n-1}) = \|T\|_{\frac{1}{3},\vec{e},\vec{\vartheta_1}} (x_1,\, x_2,\, \cdots,\, x_{j-1}\, ,\, 0\, ,\, x_j\, ,\, \cdots,\, x_{n-1})$
$= T\, (\, \frac{1}{3} \cdot [e_1 + v_1 \cdot x_1,\, \cdots, e_{j-1} + v_{j-1} \cdot x_{j-1},\, 2\, ,\, e_{j+1} + v_{j+1} \cdot x_j,\, \cdots, e_n + v_n \cdot x_{n-1}\,]\,)$
and $\left\langle \|T\|_{\frac{1}{3},\vec{e},\vec{\vartheta_{-1}}} \right\rangle_{0,j} (x_1,\, x_2,\, \cdots,\, x_{n-1}) = \|T\|_{\frac{1}{3},\vec{e},\vec{\vartheta_{-1}}} (x_1,\, x_2,\, \cdots,\, x_{j-1}\, ,\, 0\, ,\, x_j\, ,\, \cdots,\, x_{n-1})$
$= T\, (\, \frac{1}{3} \cdot [e_1 + v_1 \cdot x_1,\, \cdots\, \cdots\, ,\, 2\, ,\, \cdots\, \cdots\, , e_n + v_n \cdot x_{n-1}\,]\,)$.
Hence

$$a \cdot (\, -\prod_{v_i\in\vec{\vartheta_1}} v_i\,) \cdot \left\langle \|T\|_{\frac{1}{3},\vec{e},\vec{\vartheta_1}} \right\rangle_{0,j} + a \cdot (\, -\prod_{v_i\in\vec{\vartheta_{-1}}} v_i\,) \cdot \left\langle \|T\|_{\frac{1}{3},\vec{e},\vec{\vartheta_{-1}}} \right\rangle_{0,j} = 0 \qquad (10)$$

The same way, we get with
$\vec{e_0} := (e_1,\, e_2,\, \cdots,\, e_{j-1},\, 0,\, e_{j+1},\, \cdots,\, e_n)$, $\vec{e_2} := (e_1,\, e_2,\, \cdots,\, e_{j-1},\, 2,\, e_{j+1},\, \cdots,\, e_n) \in \mathcal{E}^n$,

$\vec{\vartheta_1} \ := (v_1,\, v_2,\, \cdots ,\, v_{j-1},\ \ 1,\, v_{j+1},\, \cdots ,\, v_n)\ ,$
$\vec{\vartheta_{-1}} := (v_1,\, v_2,\, \cdots ,\, v_{j-1},\, -1,\, v_{j+1},\, \cdots ,\, v_n) \in \widehat{\mathcal{V}}_n\ ,$ that
$\left\langle \|T\|_{\frac{1}{3},\vec{e_0},\vec{\vartheta_1}} \right\rangle_{1,j} (x_1,\, x_2,\, \cdots ,\, x_{n-1}) = \|T\|_{\frac{1}{3},\vec{e_0},\vec{\vartheta_1}} (x_1,\, x_2,\, \cdots ,\, x_{j-1}\ ,\ 1\ ,\ x_j\ ,\, \cdots ,\, x_{n-1})$
$= T\ (\ \frac{1}{3} \cdot [e_1 + v_1 \cdot x_1,\, \cdots\ \cdots ,\ 1\ ,\ \cdots\ \cdots ,e_n + v_n \cdot x_{n-1}\ \ ]\ )$
$= \|T\|_{\frac{1}{3},\vec{e_2},\vec{\vartheta_{-1}}} (x_1,\, x_2,\, \cdots ,\, x_{j-1}\ ,\ 1\ ,\ x_j\ ,\, \cdots ,\, x_{n-1})$
$= \left\langle \|T\|_{\frac{1}{3},\vec{e_2},\vec{\vartheta_{-1}}} \right\rangle_{1,j} (x_1,\, x_2,\, \cdots ,\, x_{n-1})\ .$
Hence

$$b \cdot (\ - \prod_{v_i \in \vec{\vartheta_1}} v_i\ )\ \cdot\ \left\langle \|T\|_{\frac{1}{3},\vec{e_0},\vec{\vartheta_1}} \right\rangle_{1,j} + b \cdot (\ - \prod_{v_i \in \vec{\vartheta_{-1}}} v_i\ )\ \cdot\ \left\langle \|T\|_{\frac{1}{3},\vec{e_2},\vec{\vartheta_{-1}}} \right\rangle_{1,j}\ =\ 0 \tag{11}$$

Now let's define
$\vec{e} := (e_1,\, e_2,\, \cdots ,\, e_{j-1},\, 0,\, e_{j+1},\, \cdots ,\, e_n) \in \mathcal{E}^n\ ,\ \widetilde{e} := (e_1,\, e_2,\, \cdots ,\, e_{j-1}, e_{j+1},\, \cdots ,\, e_n) \in \mathcal{E}^{n-1}\ ,$
$\vec{\vartheta}\ \ := (v_1,\, v_2,\, \cdots ,\, v_{j-1},\ \ 1,\, v_{j+1},\, \cdots ,\, v_n) \in \widehat{\mathcal{V}}_n\ ,$ and
$\widetilde{\vartheta}\ \ := (v_1,\, v_2,\, \cdots ,\, v_{j-1},\, v_{j+1},\, \cdots ,\, v_n)\ , \in \widehat{\mathcal{V}}_{n-1}\ .$
We have that $\left\langle \|T\|_{\frac{1}{3},\vec{e},\vec{\vartheta}} \right\rangle_{0,j}\ ,\ \ \left\| \langle T \rangle_{0,j} \right\|_{\frac{1}{3},\widetilde{e},\widetilde{\vartheta}} \in \mathcal{CC}_{n-1,X}$
and for all $(x_1,\, x_2,\, \cdots ,\, x_{n-1}) \in W^{n-1}$ we calculate
$\left\langle \|T\|_{\frac{1}{3},\vec{e},\vec{\vartheta}} \right\rangle_{0,j} (x_1,\, x_2,\, \cdots ,\, x_{n-1}) = \|T\|_{\frac{1}{3},\vec{e},\vec{\vartheta}} (x_1,\, x_2,\, \cdots ,\, x_{j-1}, 0,\, x_j,\, \cdots ,\, x_{n-1})$
$= T\ (\ \frac{1}{3}\ \cdot [e_1 + v_1 \cdot x_1,\, \cdots\ \cdots ,\ \ e_{j-1} + v_{j-1} \cdot x_{j-1}\, , 0\, ,\ e_{j+1} + v_{j+1} \cdot x_j,\, \cdots\ \cdots ,e_n + v_n \cdot x_{n-1}\ \ ]\ )$
$= \langle T \rangle_{0,j}\ (\ \frac{1}{3}\ \cdot [e_1 + v_1 \cdot x_1,\, \cdots\ \cdots ,\ e_{j-1} + v_{j-1} \cdot x_{j-1}\, ,\, e_{j+1} + v_{j+1} \cdot x_j,\, \cdots\ \cdots ,e_n + v_n \cdot x_{n-1}\ \ ]\ )$
$= \left\| \langle T \rangle_{0,j} \right\|_{\frac{1}{3},\widetilde{e},\widetilde{\vartheta}} (x_1,\, x_2,\, \cdots ,\, x_{j-1},\, x_j,\, \cdots ,\, x_{n-1})$
Hence ,

$$a \cdot (\ - \prod_{v_i \in \vec{\vartheta}} v_i\ )\ \cdot\ \left\langle \|T\|_{\frac{1}{3},\vec{e},\vec{\vartheta}} \right\rangle_{0,j}\ =\ a \cdot (\ - \prod_{v_i \in \widetilde{\vartheta}} v_i\ )\ \cdot\ \left\| \langle T \rangle_{0,j} \right\|_{\frac{1}{3},\widetilde{e},\widetilde{\vartheta}} \tag{12}$$

The same way , for
$\vec{e} := (e_1,\, e_2,\, \cdots ,\, e_{j-1},\, 2,\, e_{j+1},\, \cdots ,\, e_n) \in \mathcal{E}^n\ ,\ \widetilde{e} := (e_1,\, e_2,\, \cdots ,\, e_{j-1}, e_{j+1},\, \cdots ,\, e_n) \in \mathcal{E}^{n-1}\ ,$
$\vec{\vartheta}\ \ := (v_1,\, v_2,\, \cdots ,\, v_{j-1},\ \ 1,\, v_{j+1},\, \cdots ,\, v_n) \in \widehat{\mathcal{V}}_n\ ,$ and
$\widetilde{\vartheta}\ \ := (v_1,\, v_2,\, \cdots ,\, v_{j-1},\, v_{j+1},\, \cdots ,\, v_n)\ , \in \widehat{\mathcal{V}}_{n-1}\ ,$
it is $\left\langle \|T\|_{\frac{1}{3},\vec{e},\vec{\vartheta}} \right\rangle_{1,j}\ ,\ \ \left\| \langle T \rangle_{1,j} \right\|_{\frac{1}{3},\widetilde{e},\widetilde{\vartheta}} \in \mathcal{CC}_{n-1,X}\ ,$
and for an arbitrary tuple $(x_1,\, x_2,\, \cdots ,\, x_{n-1}) \in W^{n-1}$ we compute
$\left\langle \|T\|_{\frac{1}{3},\vec{e},\vec{\vartheta}} \right\rangle_{1,j} (x_1,\, x_2,\, \cdots ,\, x_{n-1}) = \|T\|_{\frac{1}{3},\vec{e},\vec{\vartheta}} (x_1,\, x_2,\, \cdots ,\, x_{j-1}, 1,\, x_j,\, \cdots ,\, x_{n-1})$
$= T\ (\ \frac{1}{3}\ \cdot [e_1 + v_1 \cdot x_1,\, \cdots\ \cdots ,\ \ e_{j-1} + v_{j-1} \cdot x_{j-1}\, , 3\, ,\ e_{j+1} + v_{j+1} \cdot x_j,\, \cdots\ \cdots ,e_n + v_n \cdot x_{n-1}\ \ ]\ )$
$= \langle T \rangle_{1,j}\ (\ \frac{1}{3}\ \cdot [e_1 + v_1 \cdot x_1,\, \cdots\ \cdots ,\ e_{j-1} + v_{j-1} \cdot x_{j-1}\, ,\, e_{j+1} + v_{j+1} \cdot x_j,\, \cdots\ \cdots ,e_n + v_n \cdot x_{n-1}\ \ ]\ )$
$= \left\| \langle T \rangle_{1,j} \right\|_{\frac{1}{3},\widetilde{e},\widetilde{\vartheta}} (x_1,\, x_2,\, \cdots ,\, x_{j-1},\, x_j,\, \cdots ,\, x_{n-1})\ .$
Hence we get

$$b \cdot (\ - \prod_{v_i \in \vec{\vartheta}} v_i\ )\ \cdot\ \left\langle \|T\|_{\frac{1}{3},\vec{e},\vec{\vartheta}} \right\rangle_{1,j}\ =\ b \cdot (\ - \prod_{v_i \in \widetilde{\vartheta}} v_i\ )\ \cdot\ \left\| \langle T \rangle_{1,j} \right\|_{\frac{1}{3},\widetilde{e},\widetilde{\vartheta}}\ \ . \tag{13}$$

This is all we need to show that $_{\vec{m}}\partial_n \circ \mathcal{SD}_n\ (T) = \mathcal{SD}_{n-1} \circ\ _{\vec{m}}\partial_n\ (T)\ ,$
and the lemma has been proved. Qed. □

Because of the Lemma, the map $\mathcal{SD}_n : \quad \Phi_{\mathcal{R},n}(X) \longrightarrow \Phi_{\mathcal{R},n}(X)$
induces a map on the space $_{\vec{m}}\mathcal{H}_n$(X) , which we also will call $\mathcal{SD}_n$.

Now we want to show that for a weight $\vec{m}$ of length 1 , that means $\vec{m} = [a, b] \in \mathcal{R}^2$, for all u $\in \Phi_{\mathcal{R},n}(X)$ with u $\in$ kernel $({}_{\vec{m}}\partial_n)$ on the level of homology classes holds :
$[\ a \cdot \mathcal{SD}_n(\mathrm{u})\ ]_\sim = [\ b \cdot \mathrm{u}\ ]_\sim$.
We are able to do this by the help of a chain homotopy in the same way we used it for proving the homotopy axiom.
This means for $n \in \mathbb{N}_{-1}$ the construction of a map $\Theta_n : \quad \Phi_{\mathcal{R},n}(X) \longrightarrow \Phi_{\mathcal{R},n+1}(X)$.
So we will get a (noncommutative) diagram :

$$\begin{array}{ccccccccc}
\dots \xrightarrow{\partial_{n+2}} & \Phi_{\mathcal{R},n+1}(X) & \xrightarrow{\partial_{n+1}} & & \Phi_{\mathcal{R},n}(X) & \xrightarrow{\partial_n} & & \Phi_{\mathcal{R},n-1}(X) & \xrightarrow{\partial_{n-1}} \dots \\
\dots & a \cdot \mathcal{SD}_{n+1} \downarrow \downarrow b \cdot Id & & \Theta_n \swarrow & a \cdot \mathcal{SD}_n \downarrow \downarrow b \cdot Id & & \Theta_{n-1} \swarrow & a \cdot \mathcal{SD}_{n-1} \downarrow \downarrow b \cdot Id & \dots \\
\dots \xrightarrow{\partial_{n+2}} & \Phi_{\mathcal{R},n+1}(X) & \xrightarrow{\partial_{n+1}} & & \Phi_{\mathcal{R},n}(X) & \xrightarrow{\partial_n} & & \Phi_{\mathcal{R},n-1}(X) & \xrightarrow{\partial_{n-1}} \dots
\end{array}$$

Our aim is to reach the equation

$$({}_{\vec{m}}\partial_{n+1} \circ \Theta_n)(\mathrm{u}) = \pm(b \cdot Id - a \cdot \mathcal{SD}_n)(\mathrm{u}) + (\Theta_{n-1} \circ {}_{\vec{m}}\partial_n)(\mathrm{u}) \ . \tag{14}$$

Of course $\Theta_{-1} := 0$, and for $n := 0$ for every T: $\{0\} \to X$ let for $x \in W$:
$\Theta_0(T)(x) := T(0)$, and we have $(\ {}_{\vec{m}}\partial_1 \circ \Theta_0)(T) = \mathrm{a} \cdot T + b \cdot T = +(b \cdot T - a \cdot \mathcal{SD}_0(T)\ )$, as required.
Let $n \geq 1$. All in all we need three auxiliary functions: $\eta_0,\ \eta_1,\ \eta_2 :\ W^2 \to W$.
Let for all $x, y \in [\ 0{,}1\ ]$: $\eta_0(x, y) := \frac{x}{3-2y}$

$$\eta_1(x, y) := \begin{cases} \frac{2-x}{3-2y} & \Leftrightarrow \quad y \leq \frac{1}{2} + \frac{1}{2}x \\ 1 & \quad\ else \end{cases}$$

$$\eta_2(x, y) := \begin{cases} \frac{2+x}{3-2y} & \Leftrightarrow \quad y \leq \frac{1}{2} - \frac{1}{2}x \\ 1 & \quad\ else \end{cases}$$

By this definition, η_0, η_1, η_2 are continuous.
Define the set $\Upsilon := \{\ 0,\ 1,\ 2\ \}$.
Let for all tuples $\vec{z} = (z_1, z_2, \cdots, z_n) \in \Upsilon^n$ and for all $T \in \mathcal{C}C_{n,X}$ be the map $G_{\vec{z}}(T) : W^{n+1} \to X$ by defining for all elements $(\ x_1,\ \cdots\ ,\ x_n, x_{n+1}\ ) \in W^{n+1}$:
$G_{\vec{z}}(T)\ (x_1,\ \cdots\ ,\ x_n, x_{n+1}) := T\ (\ \eta_{z_1}(x_1, x_{n+1}), \eta_{z_2}(x_2, x_{n+1}),\ \cdots\ ,\ \eta_{z_n}(x_n, x_{n+1})\ )$.
Thus, $G_{\vec{z}}(T)$ is out of $\mathcal{C}C_{n+1,X}$.
Let for all $\vec{z} = (z_1, z_2, \cdots, z_n) \in \Upsilon^n$: $v_{\vec{z}} := (-1)^{\sum_{i=1}^n z_i} = (-1)^{the\ number\ of\ 1's\ in\ \vec{z}}$,
and finally :

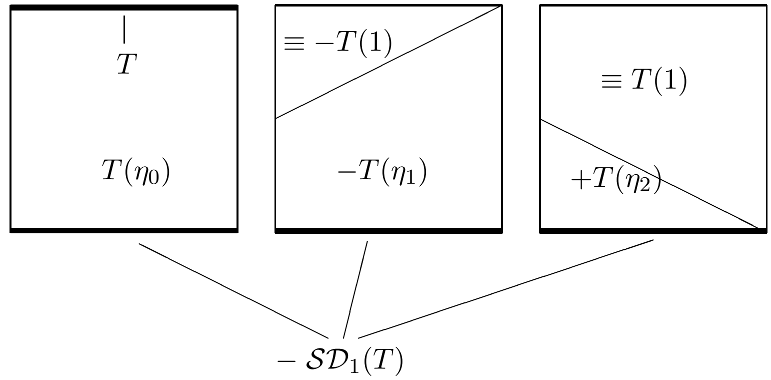
$$\Theta_n(T) \ := \sum_{\vec{z} \in \Upsilon^n} v_{\vec{z}} \cdot G_{\vec{z}}(T) \tag{15}$$

Example : For $n = 1$ for $T \in \mathcal{C}C_{1,X}$ one has for all pairs $(x, y) \in W^2$:

$$\begin{aligned}
\Theta_1(T)(x, y) &= +G_0(T)(x, y) \ - \ G_1(T)(x, y) \ + \ G_2(T)(x, y) \\
&= +T(\ \eta_0(x, y)\ ) \ - \ T(\ \eta_1(x, y)\ ) \ + \ T(\ \eta_2(x, y)\ )
\end{aligned}$$

(See the picture):

$\Theta_1(T) \quad =$



It's easy to define something, but it's hard to prove the desired result . − − −
So let's begin: Let $T \in \mathcal{C}C_{n,X}$. We have
$(\ {}_{\vec{m}}\partial_{n+1} \circ \Theta_n \ )(T) = {}_{\vec{m}}\partial_{n+1} \ (\ \sum_{\vec{z} \in \Upsilon^n} \quad v_{\vec{z}} \cdot G_{\vec{z}}(T) \ )$
$= \sum_{j=1}^{n+1} (-1)^{j+1} \quad \sum_{\vec{z} \in \Upsilon^n} \ v_{\vec{z}} \cdot (\ \ a \cdot \langle \, G_{\vec{z}}(T) \, \rangle_{0,j} \ + \ b \cdot \langle \, G_{\vec{z}}(T) \, \rangle_{1,j} \ \ ) \ .$

Now let's take $j := n+1$.
For $\vec{z} := (\ 0{,}0{,}0, \ \cdots \ , 0 \ ) \ \in \Upsilon^n$ for $(x_1, \cdots, x_n \ ) \in W^n$ we compute ('above') :
$\langle \, G_{\vec{z}}(T) \, \rangle_{1,n+1} \ (x_1, x_2, \cdots, x_n \ ) = G_{\vec{z}}(T)(x_1, \cdots, x_n, 1)$
$= T \ (\ \eta_0(x_1,1), \eta_0(x_2,1), \ \cdots \ , \ \eta_0(x_n,1) \ ) = T \ (\ x_1, x_2, \ \cdots \ , \ x_n \ ) \ .$
Hence, $b \cdot \langle \, G_{\vec{z}}(T) \, \rangle_{1,n+1} = b \cdot T$.
Let for $k \ \in \ \{1, 2, \cdots, n\}$:
$\vec{\zeta_1} := (z_1, z_2, \cdots, z_{k-1}, 1, z_{k+1}, \cdots, z_n \ ) \ , \ \ \vec{\zeta_2} := (z_1, z_2, \cdots, z_{k-1}, 2, z_{k+1}, \cdots, z_n \ ) \in \Upsilon^n$,
and for an element $(x_1, \cdots, x_n \ )$ out of W^n we get
$\left\langle \ G_{\vec{\zeta_1}}(T) \ \right\rangle_{1,n+1} (x_1, x_2, \cdots, x_n \ ) = G_{\vec{\zeta_1}}(T)(x_1, x_2, \cdots, x_n, 1 \ )$
$= T \ (\ \cdots\cdots \ , \ \eta_1(x_k, 1) \ , \cdots \ \cdots) = T \ (\ \cdots\cdots \ , \ 1 \ , \cdots \ \cdots)$
$= T \ (\ \cdots\cdots \ , \ \eta_2(x_k, 1) \ , \cdots \ \cdots) = G_{\vec{\zeta_2}}(T)(x_1, x_2, \cdots, x_n, 1 \ ) = \left\langle \ G_{\vec{\zeta_2}}(T) \ \right\rangle_{1,n+1} (x_1, \cdots, x_n \ )$
Because of $v_{\vec{\zeta_1}} \cdot v_{\vec{\zeta_2}} = -1$ we have
$b \cdot v_{\vec{\zeta_1}} \cdot \left\langle \ G_{\vec{\zeta_1}}(T) \ \right\rangle_{1,n+1} + b \cdot v_{\vec{\zeta_2}} \cdot \left\langle \ G_{\vec{\zeta_2}}(T) \ \right\rangle_{1,n+1} = 0 \ .$

Now we compute ('down') : (We still have $j := n+1$).
It holds for all $(x_1, \cdots, x_n \ ) \in W^n$ and for all $\vec{z} = (z_1, \cdots, z_n \ ) \in \Upsilon^n$:
$\langle \, G_{\vec{z}}(T) \, \rangle_{0,n+1} \ (x_1, x_2, \cdots, x_n \ ) = G_{\vec{z}}(T) \ (x_1, x_2, \cdots, x_n, \ 0 \ )$
$= \ T \ (\ \eta_{z_1}(x_1, 0), \eta_{z_2}(x_2, 0), \ \cdots \ , \ \eta_{z_n}(x_n, 0) \ ) \ =: \ T \ (\ t_1, t_2, \ \cdots \ , t_n \ \ )$
with for all $i = 1, 2, \cdots, n$:

$$t_i := \begin{cases} \frac{1}{3} \cdot x_i & \Leftrightarrow \quad z_i = 0 \\ \frac{1}{3} \cdot (2 - x_i) & \Leftrightarrow \quad z_i = 1 \\ \frac{1}{3} \cdot (2 + x_i) & \Leftrightarrow \quad z_i = 2 \end{cases}$$

Hence, with $\vec{e} := (e_1, e_2, \ \cdots \ , e_n \ ) \ , \ \vec{v} := (v_1, v_2, \ \cdots \ , v_n \ \ )$ and for $i \in \{1, 2, \cdots n\}$:

$$e_i := \begin{cases} 0 & \Leftrightarrow \quad z_i = 0 \\ 2 & \Leftrightarrow \quad z_i \in \{1, 2\} \end{cases}$$

and

$$v_i := \begin{cases} 1 & \Leftrightarrow \quad z_i \in \{0, 2\} \\ -1 & \Leftrightarrow \quad z_i = 1 \end{cases}$$

We get $v_{\vec{z}} \ \cdot \langle \, G_{\vec{z}}(T) \, \rangle_{0,n+1} = (\ \prod_{i=1}^n v_i \ ) \cdot \|T\|_{\frac{1}{3}, \vec{e}, \vec{v}}$
In this way we have : $\sum_{\vec{z} \in \Upsilon^n} \ v_{\vec{z}} \cdot \quad a \cdot \langle \, G_{\vec{z}}(T) \, \rangle_{0,n+1} = - \, a \cdot \mathcal{SD}_n$ (T).

All in all for $j = n+1$ holds :

$$\sum_{\vec{z} \in \Upsilon^n} v_{\vec{z}} \cdot [\, a \cdot \langle\, G_{\vec{z}}(T)\, \rangle_{0,n+1} \;+\; b \cdot \langle\, G_{\vec{z}}(T)\, \rangle_{1,n+1}\,] \;=\; b \cdot T - a \cdot \mathcal{SD}_n(T) \qquad (16)$$

For the next parts, let k be a fixed element of the set $\{\, 1, 2, 3, \cdots, n\, \}$.
Let $\vec{\zeta_0} := (z_1, z_2, \cdots, z_{k-1}, 0, z_{k+1}, \cdots, z_n\,)$, $\vec{\zeta_1} := (z_1, z_2, \cdots, z_{k-1}, 1, z_{k+1}, \cdots, z_n\,) \in \Upsilon^n$,
and for all elements $(x_1, \cdots, x_n\,) \in W^n$ we get
$\Big\langle\, G_{\vec{\zeta_0}}(T)\, \Big\rangle_{1,k} (x_1, x_2, \cdots, x_n\,) = G_{\vec{\zeta_0}}\,(T)\,(x_1, x_2, \cdots, x_{k-1}, 1, x_{k+1}, \cdots, x_n\,)$
$= T\,(\,\cdots\cdots,\; \eta_0(1, x_n)\,, \cdots\,\cdots) = T\,(\,\cdots\cdots,\; \frac{1}{3-2x_n}\,, \cdots\,\cdots)$
$= T\,(\,\cdots\cdots,\; \eta_1(1, x_n)\,, \cdots\,\cdots) = G_{\vec{\zeta_1}}\,(T)\,(x_1, x_2, \cdots, x_{k-1}, 1, x_{k+1}, \cdots, x_n\,)$
$= \Big\langle\, G_{\vec{\zeta_1}}(T)\, \Big\rangle_{1,k} (x_1, x_2, \cdots, x_n\,)$
Thus, $b \cdot v_{\vec{\zeta_0}} \cdot \Big\langle\, G_{\vec{\zeta_0}}(T)\, \Big\rangle_{1,k} + \; b \cdot v_{\vec{\zeta_1}} \cdot \Big\langle\, G_{\vec{\zeta_1}}(T)\, \Big\rangle_{1,k} = 0$.
(See the previous picture: The right side of $T(\eta_0)$ eliminates the right side of $T(\eta_1)$).
The same way , with
$\vec{\zeta_1} := (z_1, z_2, \cdots, z_{k-1}, 1, z_{k+1}, \cdots, z_n\,)$, $\vec{\zeta_2} := (z_1, z_2, \cdots, z_{k-1}, 2, z_{k+1}, \cdots, z_n\,) \in \Upsilon^n$,
it follows $\Big\langle\, G_{\vec{\zeta_1}}(T)\, \Big\rangle_{0,k} (x_1, x_2, \cdots, x_n\,)$
$= T\,(\,\cdots\cdots,\; \eta_1(0, x_n)\,, \cdots\,\cdots) = T\,(\,\cdots\cdots,\; t_k\,, \cdots\,\cdots)$
$= T\,(\,\cdots\cdots,\; \eta_2(0, x_n)\,, \cdots\,\cdots) = \Big\langle\, G_{\vec{\zeta_2}}(T)\, \Big\rangle_{0,k} (x_1, x_2, \cdots, x_n\,)$ with

$$t_k := \begin{cases} \frac{2}{3-2x_n} & \Leftrightarrow\; x_n \in [0, \frac{1}{2}] \\ 1 & \Leftrightarrow\; x_n \in [\frac{1}{2}, 1] \end{cases}$$

Thus , $a \cdot v_{\vec{\zeta_1}} \cdot \Big\langle\, G_{\vec{\zeta_1}}(T)\, \Big\rangle_{0,k} + \; a \cdot v_{\vec{\zeta_2}} \cdot \Big\langle\, G_{\vec{\zeta_2}}(T)\, \Big\rangle_{0,k} = 0$.
(See the previous picture again: The left side of $T(\eta_1)$ eliminates the left side of $T(\eta_2)$).
And once more with
$\vec{\zeta_0} = (z_1, z_2, \cdots, z_{k-1}, 0, z_{k+1}, \cdots, z_n\,) \in \Upsilon^n$, and $\vec{\mu} := (z_1, z_2, \cdots, z_{k-1}, z_{k+1}, \cdots, z_n\,) \in \Upsilon^{n-1}$
we get for $(x_1, \cdots, x_n\,) \in W^n$:
$\Big\langle\, G_{\vec{\zeta_0}}(T)\, \Big\rangle_{0,k} (x_1, \cdots, x_n\,) = G_{\vec{\zeta_0}}(T)(x_1, \cdots, x_{k-1}, 0, x_k, \cdots, x_n\,)$
$= T\,(\; \eta_{z_1}(x_1, x_n),\; \cdots,\; \eta_{z_{k-1}}(x_{k-1}, x_n),\; \eta_0(0, x_n),\; \eta_{z_{k+1}}(x_k, x_n),\; \cdots,\; \eta_{z_n}(x_{n-1}, x_n)\,)$
$= T\,(\;\cdots\;\;\cdots, \eta_{z_{k-1}}(x_{k-1}, x_n),\; 0,\; \eta_{z_{k+1}}(x_k, x_n),\; \cdots\;\cdots,\;)$
$= \langle\, T\,\rangle_{0,k}\,(\;\cdots\;\;\cdots,\; \eta_{z_{k-1}}(x_{k-1}, x_n),\; \eta_{z_{k+1}}(x_k, x_n),\; \cdots\;\cdots\;)$
$= G_{\vec{\mu}}\,(\,\langle\, T\,\rangle_{0,k}\,)\,(x_1, \cdots, x_n\,)$.
Hence , $a \cdot \Big\langle\, G_{\vec{\zeta_0}}(T)\, \Big\rangle_{0,k} = a \cdot G_{\vec{\mu}}\,(\,\langle\, T\,\rangle_{0,k}\,)$
And , last not least , one can show in the same way , that for
$\vec{\zeta_2} = (z_1, z_2, \cdots, z_{k-1}, 2, z_{k+1}, \cdots, z_n\,) \in \Upsilon^n$
the equality $b \cdot \Big\langle\, G_{\vec{\zeta_2}}(T)\, \Big\rangle_{1,k} = b \cdot G_{\vec{\mu}}\,(\,\langle\, T\,\rangle_{1,k}\,)$ holds .
Now we have collected all the needed facts to prove that for every $n \in \mathbb{N}$
$(\,{}_{\vec{m}}\partial_{n+1} \circ \Theta_n\,)(T) = (-1)^{n+2} \cdot (\; b \cdot \mathrm{Id} - a \cdot \mathcal{SD}_n\,)(T) + (\;\Theta_{n-1} \circ {}_{\vec{m}}\partial_n\,)\,(T)$

It is a trivial consequence that for a chain $\mathrm{u} \in \Phi_{\mathcal{R},n}(X)$ with $({}_{\vec{m}}\partial_n)\;(\mathrm{u}) = 0$ on the level of homology classes the equation $[\;\mathrm{a}\cdot\mathcal{SD}_n(\mathrm{u})\;]_\sim = [\; b \cdot \mathrm{u}\;]_\sim$ holds.

The next step is to prove that for an arbitrary $\mathrm{u} \in \mathrm{kernel}\;({}_{\vec{m}}\partial_n)$ on the level of homology classes holds that
$[\; b \cdot \mathcal{SD}_n\;(\mathrm{u})\;\;]_\sim = [\; a \cdot \mathrm{u}\;]_\sim$.

Looking on the previous proof this seems obvious, and we will not display it in all details. The proof is nearly the same, we only have to modify it by 'turning it upside down'.
Instead of using the three auxiliary functions: $\eta_0,\ \eta_1,\ \eta_2$,
we need three others $\widetilde{\eta_0},\ \widetilde{\eta_1},\ \widetilde{\eta_2}:\ W^2 \to W$.
Let for $x,y \in [\,0{,}1\,]$: $\widetilde{\eta_0}(x,y) := \frac{x}{1+2y}$

$$\widetilde{\eta_1}(x,y) := \begin{cases} \frac{2-x}{1+2y} & \Leftrightarrow \quad y \geq \frac{1}{2} - \frac{1}{2}x \\ 1 & else \end{cases}$$

$$\widetilde{\eta_2}(x,y) := \begin{cases} \frac{2+x}{1+2y} & \Leftrightarrow \quad y \geq \frac{1}{2} + \frac{1}{2}x \\ 1 & else \end{cases}$$

By this definition, $\widetilde{\eta_0}, \widetilde{\eta_1}, \widetilde{\eta_2}$ are continuous .
Let for a fixed tuple $\vec{z} = (z_1, z_2, \cdots, z_n) \in \Upsilon^n$ for $T \in \mathcal{CC}_{n,X}$ the map
$\widetilde{G_{\vec{z}}}(\mathrm{T}) : \mathrm{W}^{n+1} \to \mathrm{X}$ by defining for all $(\ x_1, \cdots, x_n, x_{n+1}\ ) \in \mathrm{W}^{n+1}$:
$\widetilde{G_{\vec{z}}}(\mathrm{T})\ (x_1,\ \cdots,\ x_{n+1}) := \mathrm{T}\ (\ \widetilde{\eta_{z_1}}(x_1, x_{n+1}), \widetilde{\eta_{z_2}}(x_2, x_{n+1}),\ \cdots,\ \widetilde{\eta_{z_n}}(x_n, x_{n+1})\ )$
Thus , $\widetilde{G_{\vec{z}}}(\mathrm{T}) \in \mathcal{CC}_{n+1,X}$.
Let for $\vec{z} = (z_1, z_2, \cdots, z_n) \in \Upsilon^n$: $v_{\vec{z}} := (-1)^{\sum_{i=1}^n z_i}$ as before ,
and finally

$$\widetilde{\Theta_n}(T) \ := \sum_{\vec{z} \in \Upsilon^n} v_{\vec{z}} \cdot \widetilde{G_{\vec{z}}}(T). \tag{17}$$

Now , by using similar calculations as above , we reach the equation
$(\ {}_{\vec{m}}\partial_{n+1} \circ \widetilde{\Theta_n}\ )(T) = (-1)^{n+2} \cdot (\ a \cdot \mathrm{Id} - b \cdot \mathcal{SD}_n\ )\ (T) + (\ \widetilde{\Theta_{n-1}} \circ {}_{\vec{m}}\partial_n\ )\ (T)$,
and , by using a cycle u (that means ${}_{\vec{m}}\partial_n$ (u) $= 0$) instead of the map T ,
it leads directly to the desired formula $[\ b\ \cdot \mathcal{SD}_n\ (\mathrm{u})\ ]_\sim = [\ a\ \cdot \mathrm{u}\ ]_\sim$.

Now let for all $k \in \mathbb{N},\ n \in \mathbb{N}_0$ and for $\mathrm{u} \in \Phi_{\mathcal{R},n}\ (X)$:
$\mathcal{SD}_n^{(k)}\ (\mathrm{u}) := (\ \mathcal{SD}_n \circ \mathcal{SD}_n \circ \cdots \circ \mathcal{SD}_n\ )\ \ (\mathrm{u}),$
(with k factors $\mathcal{SD}_n$) .

Lemma 3. *For all* $k \in \mathbb{N}$ *and* $n \in \mathbb{N}_0$ *for all* $\mathrm{u} \in$ *kernel* $({}_{\vec{m}}\partial_n)$ *it holds that*
$[\ a^k\ \cdot \mathcal{SD}_n^{(k)}\ \ (\ \mathrm{u}\ )\ \ ]_\sim = [\ b^k\ \cdot \mathrm{u}\ ]_\sim$ *and*
$[\ b^k\ \cdot \mathcal{SD}_n^{(k)}\ \ (\ \mathrm{u}\ )\ \ ]_\sim = [\ a^k\ \cdot \mathrm{u}\ ]_\sim$.

We prove the first equation by induction.
Note, that if ${}_{\vec{m}}\partial_n$ (u) $= 0$, also ${}_{\vec{m}}\partial_n\ (\mathcal{SD}_n^{(k)}(\mathrm{u})) = 0$ (because $\mathcal{SD}_n$ commutes with the boundary operator), and note that $\mathcal{SD}_n$ is a linear map.
Let for one $k \in \mathbb{N}$ for an $\mathrm{u} \in$ kernel $({}_{\vec{m}}\partial_n)$: $[\ a^k\ \cdot \mathcal{SD}_n^{(k)}\ (\mathrm{u})\ \ ]_\sim = [\ b^k \cdot \mathrm{u}\ ]_\sim$.
And let $\mathrm{w} := a^k\ \cdot \mathcal{SD}_n^{(k)}(\mathrm{u})$, hence $\mathrm{w} \in$ kernel ${}_{\vec{m}}\partial_n$.
So we get $[\ b^{k+1} \cdot \mathrm{u}\ ]_\sim = [b\ \cdot\ b^k \cdot \mathrm{u}\ ]_\sim = [b \cdot a^k\ \cdot \mathcal{SD}_n^{(k)}(\mathrm{u})\ ]_\sim = [\ b \cdot \mathrm{w}\ ]_\sim$
$= [\ a\ \cdot\ \mathcal{SD}_n(\mathrm{w})\ ]_\sim = [a \cdot a^k\ \cdot \mathcal{SD}_n\ (\mathcal{SD}_n^{(k)}(\mathrm{u})\,)\ ]_\sim = [a^{k+1}\ \cdot \mathcal{SD}_n^{(k+1)}(\mathrm{u})\ ]_\sim$,
and the lemma has been proved.

Lemma 4. *For all weights* $\vec{m} := [a,b]\ \in \mathcal{R}^2$ *with* $\{\ a,b\ \}$ *have the condition* $\mathcal{NCD}$
for all $k \in \mathbb{N}$ *holds :*
There is $r_k \in \mathcal{R}$ *such that for all* $n \in \mathbb{N}_0$ *and for all* $\mathrm{u} \in$ *kernel* $({}_{\vec{m}}\partial_n)$ *one has:*
$r_k\ \cdot [\ \mathcal{SD}_n^{(k)}\ \ (\mathrm{u})\ \ \ ]_\sim = [\ \ \mathrm{u}\ \ ]_\sim\ \ \in {}_{\vec{m}}\mathcal{H}_n(X)$.

Proof. The condition $\mathcal{NCD}$ means, that for $k \in \mathbb{N}$ exist x_k , $y_k \in \mathcal{R}$ such that
$x_k \cdot a^k + y_k \cdot b^k = 1$.
Take this condition and the previous lemma and write :

$[\ \mathrm{u}\ ]_\sim = [\ (x_k \cdot a^k + y_k \cdot b^k) \cdot \mathrm{u}\ ]_\sim = \ x_k \cdot [\ a^k \cdot \mathrm{u}\ ]_\sim + y_k \cdot [\ b^k \cdot \mathrm{u}\ ]_\sim$
$= \ x_k \cdot [\ b^k \ \cdot \ \mathcal{SD}_n^{(k)}\ (\mathrm{u})\ ]_\sim + y_k \cdot [\ a^k \ \cdot \ \mathcal{SD}_n^{(k)}\ (\mathrm{u})\ ]_\sim$
$= \ (\ x_k \cdot b^k \ + y_k \cdot \ a^k\ ) \ \cdot \ [\ \mathcal{SD}_n^{(k)}\ (\mathrm{u})\ ]_\sim \ = \ r_k \cdot \ [\ \mathcal{SD}_n^{(k)}\ (\mathrm{u})\ ]_\sim$,
with $\quad r_k := (\ x_k \cdot b^k + y_k \cdot \ a^k\ )$, qed. □

Now let's return to our purpose. We wanted to prove that the inclusion $\Phi_{\mathcal{R},n}(X,\mathcal{U}) \overset{j}{\hookrightarrow} \Phi_{\mathcal{R},n}(X)$ leads to isomorphisms j_*: ${}_{\vec{m}}\mathcal{H}_n(X,\,\mathcal{U}) \xrightarrow{\simeq} {}_{\vec{m}}\mathcal{H}_n(X\,)$, induced by j.
Note that for a continuous T: $W^n \to X$ the image $T(W^n)$ is compact in X , and for the given open covering $\mathcal{U} = \{U_i \mid i \in I\}$ of X a finite subset is sufficient to cover $T(W^n)$.
Hence, by iterating $\mathcal{SD}_n$, there is a number $k_T \in \mathbb{N}$ such that $\mathcal{SD}_n^{(k_T)}(T) \in \Phi_{\mathcal{R},n}(X,\mathcal{U})$.
And also for an u $\in \Phi_{\mathcal{R},n}(X)$ exists $k_u \in \mathbb{N}$ such that $\mathcal{SD}_n^{(k_u)}(\mathrm{u}) \in \Phi_{\mathcal{R},n}(X,\mathcal{U})$.

Let us look on the inclusion $\quad j:\ \Phi_{\mathcal{R},n}(X,\mathcal{U}) \hookrightarrow \Phi_{\mathcal{R},n}(X)$,
and the induced $\mathcal{R}$-Modul homomorphism : $\quad j_*:\ {}_{\vec{m}}\mathcal{H}_n(X,\,\mathcal{U}) \longrightarrow {}_{\vec{m}}\mathcal{H}_n(X\,)$.
We have to show that j_* is an epimorphism and monomorphism. (compare [1] , p. 36).
j_* is an epimorphism :
Let $\ \mathrm{z} \in {}_{\vec{m}}\mathcal{H}_n(X\,) \implies$ there is $\ \mathrm{u} \in \mathrm{kernel}\ ({}_{\vec{m}}\partial_n) \subset \Phi_{\mathcal{R},n}(X)$ and $[\ \mathrm{u}\ ]_\sim = \mathrm{z}$.
$\implies$ there is $\ k \in \mathbb{N}$ and $\ \mathcal{SD}_n^{(k)}(\mathrm{u}) \in \Phi_{\mathcal{R},n}(X,\mathcal{U})$.
Take the factor $\ r_k \in \mathcal{R}$ (out of Lemma 4) and
$[\ r_k \ \cdot \mathcal{SD}_n^{(k)}\ (\mathrm{u})\ ]_\sim = [\ \mathrm{u}\ ]_\sim \ \in {}_{\vec{m}}\mathcal{H}_n(X)$ and $\ r_k \cdot \mathcal{SD}_n^{(k)}(\mathrm{u}) \in \Phi_{\mathcal{R},n}(X,\mathcal{U}) \subset \Phi_{\mathcal{R},n}(X)$.
Hence, $\quad j_*\ (\ [\ r_k \ \cdot \mathcal{SD}_n^{(k)}\ (\mathrm{u})\ ]_\sim\ ) = [\ j\ (\ r_k \ \cdot \mathcal{SD}_n^{(k)}\ (\mathrm{u})\ )\ ]_\sim = [\ \mathrm{u}\ ]_\sim = \mathrm{z}$.
j_* is a monomorphism :
Let $\ \mathrm{x} \in {}_{\vec{m}}\mathcal{H}_n(X,\,\mathcal{U})$ with $j_*\ (\ \mathrm{x}) = 0$. We must show that $\ \mathrm{x} = 0$.
An element $\mathrm{x} \in {}_{\vec{m}}\mathcal{H}_n(X,\,\mathcal{U})$ means there exists a $\ v \ \in \Phi_{\mathcal{R},n}(X,\mathcal{U})$ with $[\ v\ ]_\sim = \mathrm{x}$. We must show that v is a boundary, i.e. we have to show that there is a $\ w \in \Phi_{\mathcal{R},n+1}(X,\mathcal{U})$ and $\ {}_{\vec{m}}\partial_{n+1}\ (w) = v$.
Because of $\ \mathrm{j}(v) = v$ we have $\ j_*(\ \mathrm{x}) = 0 = [\ \mathrm{j}(v)\ ]_\sim = [\ v\ ]_\sim$.
$j_*(\ \mathrm{x}) = 0 \ \in \ {}_{\vec{m}}\mathcal{H}_n(X)$ means that $\ j_*(\ \mathrm{x})$ is the equivalence class of a cycle which is a boundary, i. e. there exists a $\ \widehat{w} \in \Phi_{\mathcal{R},n+1}(X)$ and $\ {}_{\vec{m}}\partial_{n+1}\ (\widehat{w}) = v$.
Choose a sufficient large number $\ k \in \mathbb{N}$ and an element $\ r_k \in \mathcal{R}$ out of Lemma 4 such that
$[\ r_k \ \cdot \mathcal{SD}_{n+1}^{(k)}\ (\widehat{w})\ ]_\sim = [\ \widehat{w}\ ]_\sim$ with $\ \mathcal{SD}_{n+1}^{(k)}\ (\widehat{w}) \in \Phi_{\mathcal{R},n+1}(X,\mathcal{U})$, and
$[\ r_k \ \cdot \mathcal{SD}_n^{(k)}\ (v)\ ]_\sim = [\ v\ ]_\sim$ (and $\mathcal{SD}_n^{(k)}\ (v)$ is in $\Phi_{\mathcal{R},n}(X,\mathcal{U})$ by triviality) .
Hence $\quad {}_{\vec{m}}\partial_{n+1}\ (\ r_k \ \cdot \mathcal{SD}_{n+1}^{(k)}(\widehat{w})\ ) \ = \ r_k \ \cdot \ \mathcal{SD}_n^{(k)}\ ({}_{\vec{m}}\partial_{n+1}(\widehat{w})\ ) \ = \ r_k \ \cdot \ \mathcal{SD}_n^{(k)}\ (\ v\ )$.
So we conclude that $\quad r_k \ \cdot \mathcal{SD}_n^{(k)}\ (v)$ is a boundary,
and because of $\quad [\ r_k \ \cdot \mathcal{SD}_n^{(k)}\ (v)\ ]_\sim = [\ v\ ]_\sim \ \in {}_{\vec{m}}\mathcal{H}_n(X,\,\mathcal{U})$,
follows that $\quad [\ v\ ]_\sim \ = 0 = \mathrm{x}$. Qed !

Hence we have proved that $j_*:\ {}_{\vec{m}}\mathcal{H}_n(X,\,\mathcal{U}) \xrightarrow{\simeq} {}_{\vec{m}}\mathcal{H}_n(X\,)$ is an isomorphism , thus the proposition has been proved, and the proposition directly leads to the excision axiom, see again [1],30,31 .

When you read the construction of this homology theory for the first time it seems to be very difficult to compute the homolgy module of any space, except for a point . But fortunately there is an old (1968) theorem which helps us by using the ordinary singular homology theory. For $\ \mathcal{R} := \mathbb{Z}$ let for abelian groups $\mathcal{A}$ for all $n \in \mathbb{N}_0$ and for all pairs of finite

CW-Complexes (X,B): $\quad {}_{\mathcal{S}}H_n[(X,B);\mathcal{A}]$ be the n^{th} ordinary singular homology group (with coefficient group ${}_{\mathcal{S}}H_0[(point)] = \mathcal{A}$) .
Let $\vec{m} = [a,b] \in \mathbb{Z}^2$ with greatest common divisor $\{a,b\} = 1$.
Then one has for every pair of finite CW-Complexes (X,B) for all $n \in \mathbb{N}_0$:
If $\{a,b\} = \{1,-1\}$: $\quad {}_{\vec{m}}\mathcal{H}_n(X,B) = \sum_{k=0}^{n} {}_{\mathcal{S}}H_k[(X,B);\mathbb{Z}]$
If $\{a,b\} \neq \{1,-1\} \quad \Leftrightarrow \quad a+b \neq 0$:

$$
{}_{\vec{m}}\mathcal{H}_n(X,B) = \begin{cases} \sum_{k\in\mathbb{N}_0 \,\wedge\, 2k\leq n} \quad {}_{\mathcal{S}}H_{2k}[(X,B);\mathbb{Z}_{a+b}] & \Longleftrightarrow \quad n \quad even \\ \sum_{k\in\mathbb{N}_0 \,\wedge\, 2k+1\leq n} \quad {}_{\mathcal{S}}H_{2k+1}[(X,B);\mathbb{Z}_{a+b}] & \Longleftrightarrow \quad n \quad odd \end{cases}
$$

Proof: See [3], and use the homology groups of a point , computed on page 5 .

7 Dividing Through Degenerate Maps

On page 5 we computed the homology modules of a point.
Recall that $\quad \vec{m} := [m_0, m_1, m_2, \ldots, m_L]$ is the 'weight' of the homology theory,
the value $\quad \sigma := \sum_{i=0}^{L} m_i$ is the 'index' .
For $\sigma \neq 0$ we computed:

$$
{}_{\vec{m}}\mathcal{H}_n(point) = \begin{cases} 0 & \Longleftrightarrow \quad n \; odd \\ \mathcal{R}_\sigma := \mathcal{R}/(\sigma\mathcal{R}) & \Longleftrightarrow \quad n \; even \end{cases}
$$

and if $\sigma = 0$ we have for all $n \in \mathbb{N}_0$ that $\mathcal{H}_n(point) = \mathcal{R}$.
Thus 'our' homology theory ${}_{\vec{m}}\mathcal{H}_n$ differs from the usual singular homology theory.
But, as we announced in the abstract, we can divide the chain modules $\Phi_{\mathcal{R},n}(X)$ by suitable submodules and in the case $\vec{m} = [m_0, m_1]$ with $\{m_0, m_1\}$ has the condition $\mathcal{NCD}$ we obtain the usual homology theory with the coefficient module $\mathcal{R}_\sigma$.

So , as always , a new part begins with definitions:

Definition 8. *Let* $n \in \mathbb{N}$.
Let $\mathcal{DCC}_{n,X} := \{\ T \in \mathcal{CC}_{n,X} \mid$ *there is a* $j \in \{1,2,\cdots,n\}$ *and for all* $y\,,\,z \in [0,1]$ *holds :* $T(x_1, x_2, \cdots, x_{j-1}, y, x_{j+1}, \cdots, x_n\ ) = T\ (x_1, x_2, \cdots, x_{j-1}, z, x_{j+1}, \cdots, x_n\ )\ \}$, *that means that* T *does not depend on the* j^{th} *component. These* T*'s will be called 'degenerate'.*

Definition 9. *With the same conditions as in the last definition, let us define a free R-module* $\Phi\mathcal{D}_{(\mathcal{R},n)}(X)$, $\Phi\mathcal{D}_{(\mathcal{R},n)}(X) \hookrightarrow \Phi_{\mathcal{R},n}(X)$, *with :*
A chain $\sum_{i=1}^{p} r_i \cdot T_i \in \Phi_{\mathcal{R},n}(X)$ *is an element of* $\Phi\mathcal{D}_{(\mathcal{R},n)}(X)$
$\Longleftrightarrow$ *for all* $i = 1, 2, \cdots, p$ *holds that* $T_i \in \mathcal{DCC}_{n,X}$.
That means $\Phi\mathcal{D}_{(\mathcal{R},n)}(X)$ *is the free R-Modul generated by degenerate maps.*

Definition 10. *Let* $\mathcal{R}$ *be a commutative Ring with 1 , let* X *be a topological space , let* $\Phi_{\mathcal{R},n}(X)$ *be the free* $\mathcal{R}$*-Modul with the base* $\mathcal{CC}_{n,X}$ *, as before.*
Let for every $\alpha \in \mathcal{R}$: $\quad Ideal_{(\alpha,\mathcal{R},n)}(X) \hookrightarrow \Phi_{\mathcal{R},n}(X)$, *and*
$Ideal_{(\alpha,\mathcal{R},n)}(X)$ *is defined as the submodule of* $\Phi_{\mathcal{R},n}(X)$ *generated with coefficients out of* $\alpha\mathcal{R}$.
That means: A chain $\sum_{i=1}^{p} r_i \cdot T_i \in \Phi_{\mathcal{R},n}(X)$ *belongs to* $Ideal_{(\alpha,\mathcal{R},n)}(X)$
$\Longleftrightarrow$ *for all* $i = 1, 2, \cdots, p$ *holds that there is an* $y_i \in \mathcal{R}$ *such that* $r_i = y_i \cdot \alpha$.

Definition 11. *Let* $\Delta_{\sigma,\mathcal{R},n}(X) := Ideal_{(\sigma,\mathcal{R},n)}(X) + \Phi\mathcal{D}_{(\mathcal{R},n)}(X)$ *(this is not a direct sum) .*

Lemma 5. *For all weights* $\vec{m} = [m_0, m_1, \cdots, m_L]$ *the boundary operator*
${}_{\vec{m}}\partial_n : \Phi_{\mathcal{R},n}(X) \to \Phi_{\mathcal{R},n-1}(X)$ *yields a map* $\quad {}_{\vec{m}}\partial_{n\ |\Delta_{\sigma,\mathcal{R},n}(X)} : \Delta_{\sigma,\mathcal{R},n}(X) \to \Delta_{\sigma,\mathcal{R},n-1}(X)$

Proof. Let u $\in$ $\Delta_{\sigma,\mathcal{R},n}(X)$.
We have u $=$ u$_1 + u_2$ with u$_1 \in \text{Ideal}_{(\sigma,\mathcal{R},n)}(X)$, u$_2 \in \Phi\mathcal{D}_{(\mathcal{R},n)}(X)$.
$\Longrightarrow {}_{\vec{m}}\partial_n(\text{u}_1) \in \text{Ideal}_{(\sigma,\mathcal{R},n-1)}(X)$, because ${}_{\vec{m}}\partial_n$ is linear.
As well as u$_2 = \sum_{k=1}^{p} r_k \cdot T_k$, and all the $T_k's$ are degenerate.
Take $T := T_k$, and assume that T is degenerate at the $\widehat{j}^{th}$ component, $\widehat{j} \in \{ 1,2, \cdots n \}$,
and for all $y,z \in [\,0,1\,]$ we have $T\ (x_1, x_2, \cdots, x_{\widehat{j}-1}, y, x_{\widehat{j}+1}, \cdots, x_n\ )$
$= T\ (x_1, x_2, \cdots, x_{\widehat{j}-1}, z, x_{\widehat{j}+1}, \cdots, x_n\ ) =: T\ (x_1, x_2, \cdots, x_{\widehat{j}-1}, -, x_{\widehat{j}+1}, \cdots, x_n\ )$.
$\Longrightarrow\ {}_{\vec{m}}\partial_n\ (T)\ =\ \sum_{j=1}^{n} (-1)^{j+1} \sum_{i=0}^{L} m_i \cdot \langle T \rangle_{\ n,\ i,\ j}$
$= \sum_{j\in\{1,2,\cdots n\}\wedge j\neq\widehat{j}} (-1)^{j+1} \sum_{i=0}^{L} m_i\ \langle T \rangle_{\ n,\ i,\ j} \quad + \quad (-1)^{\widehat{j}+1} \cdot \sum_{i=0}^{L} m_i \cdot \langle T \rangle_{\ n,\ i,\ \widehat{j}}$.
The first sumand is a linear combination of degenerate maps.
And for an element $(\ x_1, x_2, \cdots, x_{n-1}\ ) \in W^{n-1}$ we have :
$\sum_{i=0}^{L} m_i \cdot \langle T \rangle_{\ n,\ i,\ \widehat{j}}\ (x_1, x_2, \cdots, x_{n-1}) = \sum_{i=0}^{L} m_i \cdot\ T(x_1, x_2, \cdots, x_{\widehat{j}-1}, \frac{i}{L}, x_{\widehat{j}}, x_{\widehat{j}+1}, \cdots, x_{n-1}\ )$
$= T(x_1, x_2, \cdots, x_{\widehat{j}-1}, -, x_{\widehat{j}}, x_{\widehat{j}+1}, \cdots, x_{n-1})\ \cdot\ \sum_{i=0}^{L} m_i$
$= T(x_1, x_2, \cdots, x_{\widehat{j}-1}, -, x_{\widehat{j}}, x_{\widehat{j}+1}, \cdots, x_{n-1})\ \cdot\ \sigma$
Thus, the second sumand is an element out of the $\text{Ideal}_{(\sigma,\mathcal{R},n-1)}(X)$, qed □

Definition 12. *Let for all $n \in \mathbb{N}$:* $\Phi_{\mathcal{R},n}(X)_{\sim\Delta,\sigma} := \frac{\Phi_{\mathcal{R},n}(X)}{\Delta_{\sigma,\mathcal{R},n}(X)}$

Thus the boundary operator ${}_{\vec{m}}\partial_n$ yields a chain complex :

$$\ldots \xrightarrow{{}_{\vec{m}}\partial_{n+2}} \Phi_{\mathcal{R},n+1}(X)_{\sim\Delta,\sigma} \xrightarrow{{}_{\vec{m}}\partial_{n+1}} \Phi_{\mathcal{R},n}(X)_{\sim\Delta,\sigma} \xrightarrow{{}_{\vec{m}}\partial_{n}} \Phi_{\mathcal{R},n-1}(X)_{\sim\Delta,\sigma} \xrightarrow{{}_{\vec{m}}\partial_{n-1}} \ldots$$

this leads to homology $\mathcal{R}$-modules : $\widetilde{{}_{\vec{m}}\mathcal{H}_{n/\Delta}}(X) := \frac{kernel({}_{\vec{m}}\partial_n)}{image({}_{\vec{m}}\partial_{n+1})}$.
Example: Let $\mathcal{R} := \mathbb{Z}$.
We already know, that for the one-point space $\{p\}$ we get for $\sigma \neq 0$:

$${}_{\vec{m}}\mathcal{H}_n(p) = \begin{cases} 0 & for\ n\ odd \\ \mathbb{Z}_\sigma & for\ n\ even \end{cases}$$

and $\sigma = 0 \quad \Leftrightarrow \quad$ for all $n \in \mathbb{N}_0 : \mathcal{H}_n(p) = \mathbb{Z}$.
For the space $\{p\}$ for $n \in \mathbb{N}$ the map T: $W^n \rightarrow \{p\}$ is degenerate, but T: $W^0 \rightarrow \{p\}$ is not; hence $\Delta_{\sigma,\mathbb{Z},0}(p) = \text{Ideal}_{(\sigma,\mathbb{Z},0)}(p) = \sigma \cdot \mathbb{Z}$, thus the generating chain complex
$\ldots \xrightarrow{{}_{\vec{m}}\partial_3} \Phi_2(p) \xrightarrow{{}_{\vec{m}}\partial_2} \Phi_1(p) \xrightarrow{{}_{\vec{m}}\partial_1} \Phi_0(p) \xrightarrow{{}_{\vec{m}}\partial_0} \{0\} \quad = \quad \ldots \xrightarrow{{}_{\vec{m}}\partial_3} \mathbb{Z} \xrightarrow{{}_{\vec{m}}\partial_2} \mathbb{Z} \xrightarrow{{}_{\vec{m}}\partial_1} \mathbb{Z} \xrightarrow{{}_{\vec{m}}\partial_0} \{0\}$
turns, by dividing for each n through $\Delta_{\sigma,\mathbb{Z},n}(p)$, into :

$$\ldots \xrightarrow{{}_{\vec{m}}\partial_3} \Phi_2(p)_{\sim\Delta,\sigma} \xrightarrow{{}_{\vec{m}}\partial_2} \Phi_1(p)_{\sim\Delta,\sigma} \xrightarrow{{}_{\vec{m}}\partial_1} \Phi_0(p)_{\sim\Delta,\sigma} \xrightarrow{{}_{\vec{m}}\partial_0} \{0\} \quad = \quad \ldots \xrightarrow{{}_{\vec{m}}\partial_3} \{0\} \xrightarrow{{}_{\vec{m}}\partial_2} \{0\} \xrightarrow{{}_{\vec{m}}\partial_1} \mathbb{Z}_\sigma \xrightarrow{{}_{\vec{m}}\partial_0} \{0\}$$

$$\Longrightarrow \qquad \widetilde{{}_{\vec{m}}\mathcal{H}_{n/\Delta}}(p) = \begin{cases} 0 & for\ all\ n\ \neq 0 \\ \mathbb{Z}_\sigma & for\ \ n\ = 0 \end{cases}$$

Corollary 1. *If we take the weight $\vec{m} = [m_0, m_1] \in \mathbb{Z}^2$ and if $\{m_0, m_1\}$ has the condition $\mathcal{NCD}$ (that means that the greatest common divisor of $\{\ m_0, m_1\ \}$ is 1), then holds that the homology theory $\widetilde{{}_{\vec{m}}\mathcal{H}_{n/\Delta}}$ is isomorphic to the usual singular homology theory on all pairs of finite CW-complexes, and the coefficient group is $\mathbb{Z}_{m_0+m_1}$.*

Proof. As we proved before , the homology theory ${}_{\vec{m}}\mathcal{H}$ satisfies all except one of the Eilenberg-Steenrod axioms, that means the axioms of exactness, homotopy and excision are satisfied. By dividing the chain modules with $\Delta_{\sigma,\mathcal{R},n}(X, A)$ one gets the homology modules $\widetilde{{}_{\vec{m}}\mathcal{H}_{n/\Delta}}$(X,A), and

the fourth axiom will be added, while the other three are remaining. Thus, with the uniqueness theorem proved by Eilenberg and Steenrod, we have the uniqueness of the homology groups for all finite CW-complexes (X,A). See [7] or, much easier [6], or compare [4] . $\square$

Corollary 2. *The usual singular homology theory is a special case of the class which is developed here. If we take the weight $\vec{m} := [1,-1] \in \mathbb{Z}^2$ then holds:*
The homology theory $\widetilde{{}_{\vec{m}}\mathcal{H}_{/\Delta}}$ is isomorphic to the usual singular homology theory on all pairs of finite CW-complexes, and the coefficient group is $\mathbb{Z}$.

Proof. By the previous corollary. Or see, for the last time, [1],pages 11-37 . $\square$

8 Final Remarks

We cannot decide whether this new homology theory is important or not. Perhaps it might be an interesting tool for other mathematicians. It's easy to see one difficulty: The computation of the homology-modules of the one-point-space is very simple. But to do the same for other topological spaces might be more complicate (except for finite CW-complexes, see page 21) , although the homotopy axiom and the excision axiom and the exactness axiom hold, because there are too less 0's in the homology-modules of a point, i.e. only every second is 0 .
So it would be an improvement for a better application if we can increase the number of 0's.
Let X be any toplogical space . If
$q_n : \Phi_{\mathcal{R},n}(X) \rightarrow \frac{\Phi_{\mathcal{R},n}(X)}{\Delta_{\sigma,\mathcal{R},n}(X)}$ is the canonical quotient map , then the following diagram commutes:

$$\begin{array}{ccccccccccccc}
\ldots \overset{{}_{\vec{m}}\partial_5}{\longrightarrow} & \Phi_{\mathcal{R},4}(X) & \overset{{}_{\vec{m}}\partial_4}{\longrightarrow} & \Phi_{\mathcal{R},3}(X) & \overset{{}_{\vec{m}}\partial_3}{\longrightarrow} & \Phi_{\mathcal{R},2}(X) & \overset{{}_{\vec{m}}\partial_2}{\longrightarrow} & \Phi_{\mathcal{R},1}(X) & \overset{{}_{\vec{m}}\partial_1}{\longrightarrow} & \Phi_{\mathcal{R},0}(X) & \overset{{}_{\vec{m}}\partial_0}{\longrightarrow} & \{0\} \\
\ldots & \downarrow q_4 & & \downarrow q_3 & & \downarrow q_2 & & \downarrow q_1 & & \downarrow q_0 & & \downarrow 0 \\
\ldots \overset{{}_{\vec{m}}\partial_5}{\longrightarrow} & \Phi_{\mathcal{R},4}(X)_{\sim\Delta,\sigma} & \overset{{}_{\vec{m}}\partial_4}{\longrightarrow} & \Phi_{\mathcal{R},3}(X)_{\sim\Delta,\sigma} & \overset{{}_{\vec{m}}\partial_3}{\longrightarrow} & \Phi_{\mathcal{R},2}(X)_{\sim\Delta,\sigma} & \overset{{}_{\vec{m}}\partial_2}{\longrightarrow} & \Phi_{\mathcal{R},1}(X)_{\sim\Delta,\sigma} & \overset{{}_{\vec{m}}\partial_1}{\longrightarrow} & \Phi_{\mathcal{R},0}(X)_{\sim\Delta,\sigma} & \overset{{}_{\vec{m}}\partial_0}{\longrightarrow} & \{0\}
\end{array}$$

Let β an arbitrary element out of the set $\mathbb{N}_0 \cup \{\infty\}$, then define a chain complex ${}_{\vec{m},\beta}\mathcal{K}(X)$

$$:= \quad \ldots \quad \overset{\overline{{}_{\vec{m}}\partial_4}}{\longrightarrow} \quad \overline{{}_{\beta}\Phi_{\mathcal{R},3}(X)} \quad \overset{\overline{{}_{\vec{m}}\partial_3}}{\longrightarrow} \quad \overline{{}_{\beta}\Phi_{\mathcal{R},2}(X)} \quad \overset{\overline{{}_{\vec{m}}\partial_2}}{\longrightarrow} \quad \overline{{}_{\beta}\Phi_{\mathcal{R},1}(X)} \quad \overset{\overline{{}_{\vec{m}}\partial_1}}{\longrightarrow} \quad \overline{{}_{\beta}\Phi_{\mathcal{R},0}(X)} \quad \overset{\overline{{}_{\vec{m}}\partial_0}}{\longrightarrow} \quad \{0\}$$

with for all $p \in \mathbb{N}_0$:

$$\overline{{}_{\beta}\Phi_{\mathcal{R},p}(X)} := \begin{cases} \Phi_{\mathcal{R},p}(X) & for\ p \geq \beta \\ \Phi_{\mathcal{R},p}(X)_{\sim\Delta,\sigma} & for\ 0 \leq p < \beta \end{cases}$$

For all $\beta \in \mathbb{N}_0$ we have $\overline{{}_{\vec{m}}\partial_\beta} := {}_{\vec{m}}\partial_\beta \circ q_\beta = q_{\beta-1} \circ {}_{\vec{m}}\partial_\beta$.
For the special cases $\beta = 0$ or $\beta = \infty$, we have for all $p \in \mathbb{N}_0$:
$\overline{{}_{0}\Phi_{\mathcal{R},p}(X)} = \Phi_{\mathcal{R},p}(X)$ or $\overline{{}_{\infty}\Phi_{\mathcal{R},p}(X)} = \Phi_{\mathcal{R},p}(X)_{\sim\Delta,\sigma}$ respectively.
By defining for $n \in \mathbb{N}_0$: ${}_{\vec{m},\beta}\mathcal{H}_n(\mathrm{X}) := \frac{kernel(\overline{{}_{\vec{m}}\partial_n})}{image(\overline{{}_{\vec{m}}\partial_{n+1}})}$ we have homology theories ; with the two we developed here as special cases, i.e.
${}_{\vec{m},0}\mathcal{H} = {}_{\vec{m}}\mathcal{H}$ and ${}_{\vec{m},\infty}\mathcal{H} = \widetilde{{}_{\vec{m}}\mathcal{H}_{/\Delta}}$.

Example : For the one-point space {p} (which is our favourite topological space obviously) and R $:= \mathbb{Z}$ and for the weight $\vec{m} := [-1,5]$ and for $\beta := 7$ or $\beta := 8$ we obtain for $n \in \mathbb{N}_0$

$${}_{[-1,5],7}\mathcal{H}_n(p) = \begin{cases} \mathbb{Z}_4 & for\ all\ \ n \in \{\ 0, 8, 10, 12, 14,\ \ldots\ \} \\ \mathbb{Z} & for\ n = 7 \\ 0 & for\ all\ \ n \in \{1,\ \ldots\ ,\ 6, 9, 11, 13, 15,\ \ldots\} \end{cases}$$

$$
{}_{[-1,5],8}\mathcal{H}_n(p) = \begin{cases} \mathbb{Z}_4 & for\ all \quad n \in \{\ 0, 8, 10, 12, 14,\ \dots\ \} \\ 0 & for\ all \quad n \in \{1,\ \dots\ , 7, 9, 11, 13,\ \dots\} \end{cases}
$$

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